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Kenya Climate Change Interdependence Modelling using Archimedean Copulas

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Abstract. Using Kenya historical rainfall and temperature climate data from 1991 to 2016, we conduct an analysis of the interdependence structure for climate change risk analysis. The Archimedean copula approach to dependence modelling has been used by various researchers in modelling interdependent rates. In the present study, we apply the Archimedean copulas to address interdependence effects on climate change data from a developing country. Our research encompasses several key objectives: identifying the interdependence structure of rainfall and temperature data, construction of Clayton, Frank and Gumbel copulas, fitting and selecting the optimum copulas model using maximum log-likelihood criteria and predicting future rainfall and temperature values. The method of moment procedure is used to calibrate the copula dependence parameter and maximum likelihood estimation for the marginal distribution specifications. Subsequently, the performance of the marginal distribution is compared following the AIC, BIC and Log-likelihood values. The findings show that the Frank copula with gamma marginal distribution appropriately accounts for the interdependence effects. Assuming independence of climate data leads to an underestimation of the climate change risk. This dependence structure is utilized to predict future weather patterns. The research findings have a significant effect on Kenya's economy since agriculture is a major economic activity heavily dependent on accurate weather predictions.

Key words: Archimedean copulas; statistical dependence; climate change; gamma distribution; positive stable distribution.
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Résumé (Abstract in French) L'analyse présentée utilise une approche statistique avancée pour modéliser et comprendre l'interdépendance entre les précipitations et la température, deux variables cruciales dans l'évaluation des risques liés au changement climatique. En s'appuyant sur les copules Archimédiennes, spécifiquement les copules de Clayton, Frank et Gumbel, l'étude capture les structures de dépendance complexes au sein des données climatiques historiques du Kenya, couvrant la période de 1991 à 2016. Cette méthode rigoureuse permet une représentation plus réaliste des comportements conjoints entre température et précipitations que l'hypothèse d'indépendance. Les principaux points de l'étude incluent : (a) Identification de la structure d'interdépendance : La recherche identifie comment les précipitations et la température interagissent, reconnaissant les interdépendances inhérentes qui impactent la modélisation du changement climatique; (b) Construction et sélection des copules : En construisant et ajustant les copules de Clayton, Frank et Gumbel, et en évaluant leur performance à l'aide du critère de maximum de log-vraisemblance, l'étude identifie que la copule de Frank avec une distribution marginale gamma est la plus appropriée; (c) Calibration des paramètres et évaluation des distributions marginales : L'utilisation de la méthode des moments pour calibrer les paramètres de dépendance des copules, et de l'estimation du maximum de vraisemblance pour les distributions marginales, garantit une calibration solide. (d) Les métriques telles que l'AIC, le BIC et les valeurs de log-vraisemblance permettent de comparer la performance des modèles;(d) Implications pour les risques climatiques et les prévisions : Les résultats montrent que l'ignorance des interdépendances conduit à sous-estimer les risques liés au changement climatique. (e) Le modèle optimal permet de prédire les précipitations et les températures futures, un outil essentiel pour la planification agricole et économique au Kenya; (f) Impact sur l'économie : L'agriculture étant un pilier central de l'économie kényane, l'amélioration de la précision des prévisions météorologiques grâce à cette étude a des implications significatives. Des prévisions améliorées soutiennent une meilleure allocation des ressources, la planification des cultures et les stratégies de gestion des risques; Cette recherche met en lumière l'efficacité des copules Archimédiennes pour analyser les données climatiques et leur utilité pour faire face aux défis du changement climatique dans les pays en développement.

Presentation of the author.

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1. Introduction

Copulas have been introduced in a celebrated paper by Sklar (1959) who showed that any multivariate joint distribution can be specified as a copula function of its one-dimensional marginal distributions. In the 2000s, research and applications of copula theory grew extensively. Four series of international conferences, organized in the first decade of this century further led to the growth of research in the theory of copulas. The first conference was organized in Barcelona in 2000, second in Quebec City in 2004, third in Tartu in 2007, and the fourth conference was organized in Warsaw in 2009. Pancheko (2005) developed goodness-of-fit test for copulas. The test is applied to Gaussian copula for a large US stocks dataset. He concluded that Gaussian copula is inappropriate for describing dependence among various portfolios. Patton (2009) introduces the concept of bivariate conditional copulas and showed that conditional copulas satisfy usual Sklar's theorem. Genest & Favre (2007) discussed application in hydrological engineering. Burns *et al.*,(2003) and Abbas (2006) discussed applications in data management and in decision analysis respectively. Yan (2007) discussed different kinds of copulas in R language. Schoelzel & Friederichs (2008) propose copulas approach to model non-normally distributed weather variables in climate research.

Despite the above developments, the application of copulas in many fields of sciences grew enormously in the second decade of the 21st century. Huang*et al.*, (2015) discussed application of Archimedean copulas in drought risk analysis. Burney *et al.*,(2018) discussed application of multivariate T-copula in software project management and analyse some risk factors that may lead to software project failure. Copulas have also been applied in actuarial literature to model interdependent rates (Nelsen (2007)). In the copula approach, the marginals and the association structure have to be specified (Czado *et al.*,(2012); Goethals *et al.*,(2010)) for the joint distribution function to be constructed. Onchere *et al.*,(2023) have applied Gumbel copulas to model Kenyan joint-life last survivor rates.

Agriculture being a major economic activity for most communities in Kenya is heavily dependent on the climate conditions. Extreme climate changes can greatly impact on the economic well being of the Kenyan people. This research outlines a dependence model that will help improve the modelling process of climate change data to ensure accurate predictions of the weather patterns and better disaster management strategies.

Many studies in the literature have applied multivariate and time-series methods to assess climate change risks (Wilks (2005)), others have accounted for interdependence of climate data in forecasting using copulas (AghaKouchak *et al.*,(2010),Serinaldi (2008),Laux *et al.*,(2011)). This interdependence exists in weather patterns and should be incorporated to ensure accurate predictions. There is however scanty literature of copulas applications in modelling climate change risk in developing countries climate data. Till now, there has not been a study in Kenya applying copulas to model climate change interdependence risks using rainfall and temperature data for simulation purposes. This study will help improve food security and risk mitigation measures by ensuring accurate weather predictions.

2. Methodology

Our research encompasses several key objectives: firstly, construction of Clayton, Frank and Gumbel copulas dependence functions. Secondly, fitting and selecting the optimum copulas and marginal distributions using maximum log-likelihood criteria. Finally, predicting future rainfall and temperature values using the selected optimum copulas dependence model.

2.1. Archimedean copula

The family of Archimedean copula (Cossette *et al.*,(2017); Li & Lu (2018)) is expressed via reference to a generator function as

$$C_\eta(u_1, ..., u_k) = \Phi\{\sum_{i=1}^{k} \Phi^{-1}(u_i)\}. \tag{1}$$

In the bi-variate case where u_1, u_2 represents rainfall and temperature climate data distributions respectively. We have,

$$C_\eta(u_1, u_2) = \Phi\{\Phi^{-1}(u_1) + \Phi^{-1}(u_2)\}. \tag{2}$$

η is the copula C dependence parameter. The generator function $\Phi(.)$ is any positive decreasing function and positive second derivative with $\Phi(0) = 1$ and $\Phi^{-1}(.)$ its pseudo-inverse function.

Some special cases

We discuss below two special cases where the Archimedean generator function is a Laplace transform leading to the constructions of the Clayton and Gumbel copulas dependence structures.

Example 1.

If we let $\Phi(.)$ be the Laplace transform of an identifiable gamma density, then the Clayton copula function is obtained as

$$C(u_1, u_2) = (u_1^{-\eta} + u_2^{-\eta} - 1)^{-1/\eta}. \tag{3}$$

(see proof in the Appendix 1)

Example 2.

If $\Phi(.)$ is the Laplace transform of an identifiable positive stable distribution, then the Gumbel copula model is obtained as

$$C(u_1, u_2) = \exp\{-[(-\ln u_1)^\eta + (-\ln u_2)^\eta]^{1/\eta}\}. \tag{4}$$

(see proof in the Appendix 2)

Frank copula

The frank copula models an interdependence structure where there is no upper or lower tail dependence. The copula function is given by

$$C(u_1, u_2) = -\frac{1}{\eta}\ln\{1 + \frac{(e^{-\eta u_1} - 1)(e^{-\eta u_2} - 1)}{(e^{-\eta} - 1)}\}. \tag{5}$$

2.2. Marginal distributions

We propose to apply the Weibull, logistic and gamma as the candidate marginal distributions since these represent uni-modal shapes. The better fitting distribution based on model criteria value will be selected as the model marginal distribution function.

The Weibull distribution function is given by

$$F(x) = 1 - \exp(-\beta x^\rho); \beta, \rho > 0; x \geq 0. \tag{6}$$

The logistic distribution function is expressed as

$$F(x) = 1 - \frac{1}{1 + \exp((x - m)/s)}; x > 0, m > 0, s > 0. \tag{7}$$

The logistic is applied to model intensity rate that is rising, declining or unchanged.

The gamma distribution function with shape parameter η and scale parameter γ is expressed as

$$F(x) = 1 - \frac{\Gamma(\eta, x\gamma)}{\Gamma(\eta)}; x > 0, \eta > 0, \gamma > 0. \tag{8}$$

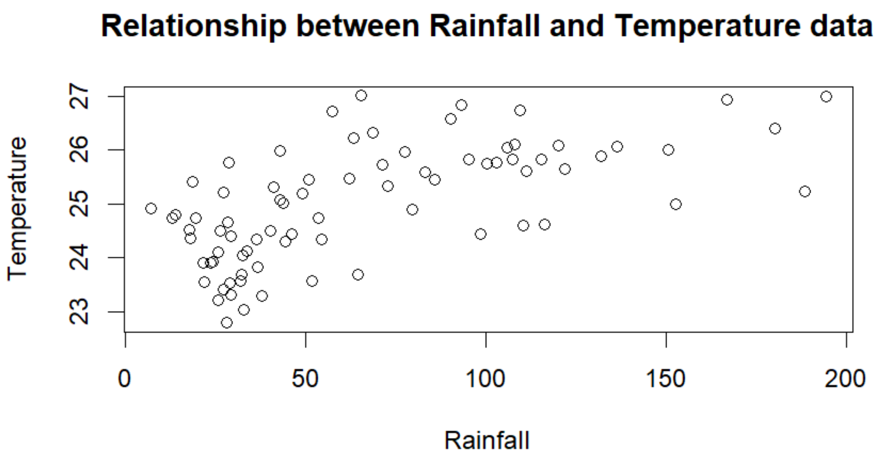
The gamma includes an additional shape specification that models bimodal data.

3. Data analysis

3.1. The Data

Kenya rainfall and temperature data for the months of April, August and December from 1991 to 2016 is used to estimate the marginal distribution parameters and to compute the level of association. The one third monthly data was used for stationarity reasons. A scatter plot of the relationship between rainfall and temperature data is shown in Figure 1. This dataset is secondary data obtained from the climate knowledge portal by world bank see (https:/ climateknowledgeportal. worldbank.org/download-data). The data contains the monthly average rainfall data measured in millimetre (mm) and temperature in degrees Celsius.

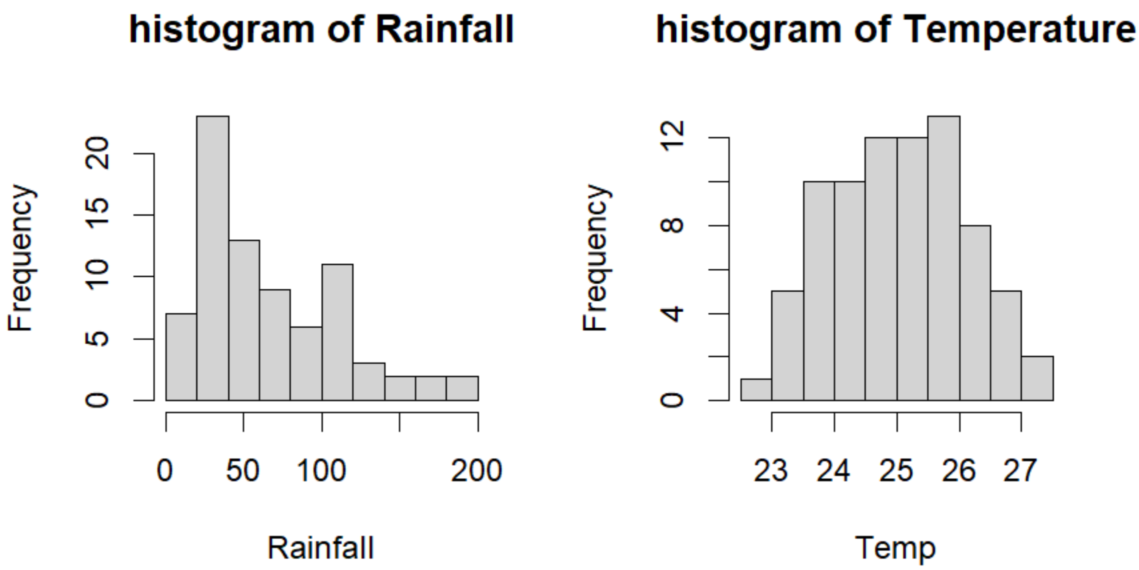
Fig. 1. Relationship between Rainfall and Temperature climate data



3.2. Descriptive statistics

The table below shows the summary statistics for the rainfall and temperature data considered in the study. The minimum rainfall amount is 6.987 mm, median 51.23 mm and maximum 194.147 mm, this huge deviation is also shown in the histogram which is skewed to the right. Whereas for the temperature data the difference is minimal i.e. minimum of 22.79 Celsius, median 25.01 Celsius and maximum 27.02 Celsius. The histogram is however slightly skewed to the left. This shows that the normality assumption is inappropriate and therefore different distributions should be applied to model the marginals as provided for in the copulas approach.

	Min.	**1st Qu.**	**Median**	**Mean**	**3rd Qu.**	**Max**
1. Rainfall	6.987	28.989	51.230	66.549	99.800	194.147
2. Temperature	22.79	24.31	25.01	25.01	25.82	27.02



Correlation tests between rainfall and temperature data at 95% confidence level shows there exists association in-fact Kendall's Tau = 0.4465534, Spearman's rho = 0.6503749 and Pearson's correlation = 0.6283184 with P-values $(7.219 \times 10^{-09}, 2.2 \times 10^{-16} \& 7.351 \times 10^{-10})$ respectively. These results are significant at $\alpha = 0.05$ leading to the rejection of the null hypotheses that true correlation is equal to zero.

3.3. Parameter estimation

For Archimedean copulas, Kendall's tau is a function of the parameters of the copula. Therefore, we can fit Archimedean copulas using the method of moments approach.

Method of moments

We would calculate Kendall's tau for the observations and equate this to the formula for Kendall's tau for the copula, solving to obtain the fitted value of η.

For the Gumbel copula, Kendall's tau is given by the formula $\tau = 1 - \frac{1}{\eta}$ $\therefore 0.4465534 = 1 - \frac{1}{\eta}$; $\eta = 1.807$

For the Clayton copula, Kendall's tau is given by the formula $\tau = \frac{\eta}{\eta+2}$ $\therefore 0.4465534 = \frac{\eta}{\eta+2}$; $\eta = 1.614$

For the Frank copula, Kendall's tau is given by the formula

$$\tau = 1 - \frac{4}{\eta}[\frac{1}{\eta}\int_0^{\eta} \frac{t}{e^t - 1}dt - 1].$$

The estimate is obtained using *copula* package in *R* as $\eta = 4.841$

Maximum-likelihood fitting

If the marginal follows the logistic distribution then the log-likelihood $\ell(m, s)$ considering a given set of climate data $\underline{t} = (t_1, t_2, \dots, t_k)$ is represented by

$$\ell(m, s|\underline{t}) = \sum_{i=1}^{k} \log(\frac{\frac{1}{s}\exp((t_i - m)/s)}{(1 + exp((t_i - m)/s))^2}). \tag{9}$$

The estimates $\hat{m}, \hat{s}$ can be derived from the non-linear equations $\frac{\partial \ell}{\partial m} = 0$ and $\frac{\partial \ell}{\partial s} = 0$ using any iterative methods. In the study we apply *MASS* package in *R* algorithms.

If the marginal follows the Weibull distribution then the log-likelihood $\ell(\beta, \rho)$ considering a given set of climate data $\underline{t} = (t_1, t_2, \dots, t_k)$ is expressed as

$$\ell(\beta, \rho|\underline{t}) = \sum_{i=1}^{k} \log(\beta\rho {t_i}^{\rho-1} \exp(-\beta {t_i}^{\rho})). \tag{10}$$

The estimates $\hat{\beta}, \hat{\rho}$ can be derived from the non-linear equations $\frac{\partial \ell}{\partial \beta} = 0$ and $\frac{\partial \ell}{\partial \rho} = 0$

If the marginal follows the gamma distribution then the log-likelihood $\ell(\gamma, \eta)$ considering a given set of climate data is given by

$$\ell(\gamma, \eta | \underline{t}) = \sum_{i=1}^{k} \log\left(\frac{1}{\Gamma(\eta)} \gamma^{\eta} {t_i}^{\eta-1} \exp\left(-\gamma t_i\right)\right). \tag{11}$$

The estimates $\hat{\gamma}, \hat{\eta}$ can be derived from the non-linear equations $\frac{\partial \ell}{\partial \gamma} = 0, \frac{\partial \ell}{\partial \eta} = 0$

Table 1. Marginal Distributions Parameter Estimates.

Marginals	*Parameter Estimates*	*AIC*	*BIC*	*LogLik*
1. Weibull	$\beta_1 = 1.543, \rho_1 = 74.33$	794.72	799.43	-395.36
	$\beta_2 = 25.92, \rho_2 = 25.52$	239.05	243.77	-117.53
2. logistic	$m_1 = 61.19, s_1 = 25.88$	821.84	826.55	-408.92
	$m_2 = 25.02, s_2 = 0.6316$	239.73	244.44	-117.86
3. gamma	$\gamma_1 = 2.199, \eta_1 = 0.033$	792.33	797.04	-394.16
	$\gamma_2 = 549.78, \eta_2 = 21.98$	234.51	239.23	-115.26

Decision: Based on the smallest AIC, BIC and largest Log-likelihood values the gamma distribution is picked to model rainfall and temperature as it gives a better fit to the climate data.

3.4. The proposed models

Having selected the gamma as the optimum marginal distribution we have three proposed models namely;

The Gumbel-gamma model which is given by

$$\begin{aligned} C(u_1, u_2) &= \exp\{-[(-\ln u_1)^{\eta} + (-\ln u_2)^{\eta}]^{1/\eta}\}, \\ &= \exp\{-[(-\ln(1 - \frac{\Gamma(\eta, \underline{t}_1 \gamma)}{\Gamma(\eta)}))^{\eta} + (-\ln(1 - \frac{\Gamma(\eta, \underline{t}_2 \gamma)}{\Gamma(\eta)}))^{\eta}]^{1/\eta}\}, \end{aligned}$$

where $\underline{t}_1$ represents the rainfall data and $\underline{t}_2$ the temperature data.

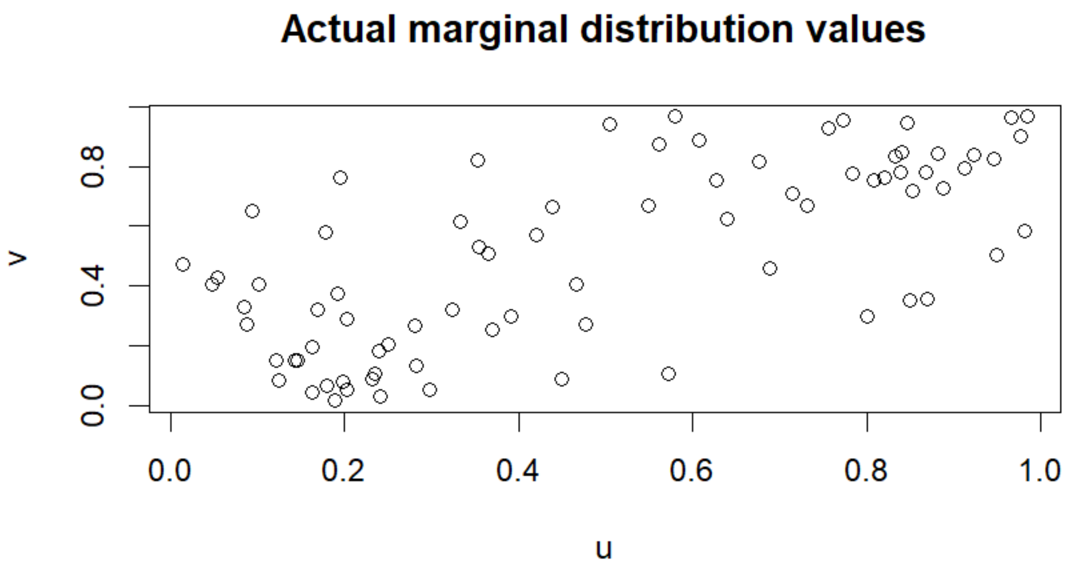
The Clayton-gamma model

$$\begin{aligned} C(u_1, u_2) &= (u_1^{-\eta} + u_2^{-\eta} - 1)^{-1/\eta}, \\ &= ((1 - \frac{\Gamma(\eta, \underline{t}_1 \gamma)}{\Gamma(\eta)})^{-\eta} + (1 - \frac{\Gamma(\eta, \underline{t}_2 \gamma)}{\Gamma(\eta)})^{-\eta} - 1)^{-1/\eta}, \end{aligned}$$

and the Frank-gamma model

$$
\begin{aligned}
C(u_1, u_2) &= -\frac{1}{\eta}\ln\{1 + \frac{(e^{-\eta u_1} - 1)(e^{-\eta u_2} - 1)}{(e^{-\eta} - 1)}\} \\
&= -\frac{1}{\eta}\ln\{1 + \frac{(e^{-\eta(1-\frac{\Gamma(\eta,t_1\gamma)}{\Gamma(\eta)})} - 1)(e^{-\eta(1-\frac{\Gamma(\eta,t_2\gamma)}{\Gamma(\eta)})} - 1)}{(e^{-\eta} - 1)}\}.
\end{aligned}
$$

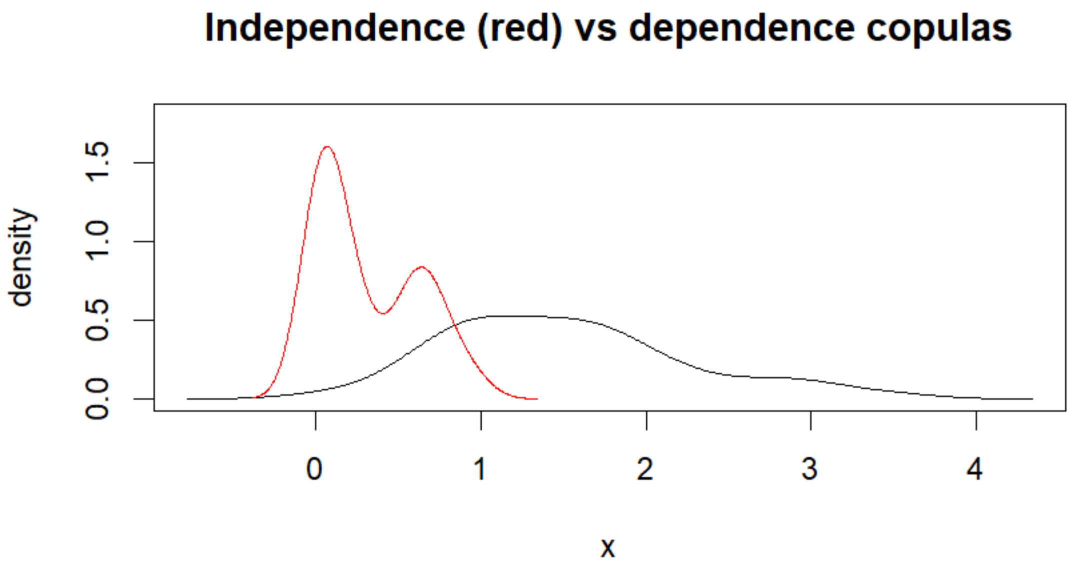
Fig. 2. Scatter plot of gamma marginal distribution functions



From the scatter plot of the gamma marginal distributions for rainfall (u) and temperature (v) datasets, we observe that there exists association in which there is no strict upper or lower tail dependence (Figure 2). There are data-points both on the top left, bottom and top right corners of the plot. The Gumbel, Clayton and Frank copula models will be fitted to the above actual distributions to pick the most appropriate dependence structure.

The Appendix 3 shows the fitted copulas in *R* using the marginals as input. The maximized log-likelihood results indicates that the Frank copula is the most appropriate model to account for the association effects compared to the Gumbel or Clayton copulas.

Fig. 3. Independence vs Dependence Copulas



A comparison plot between the independence and dependence assumptions (Figure 3), shows the independence assumption leads to an underestimation of the extreme observation points (x).

As shown in Figure 4, we observe a good fit between the actual and simulated (red points) distribution values. The two-sample Kolmogorov-Smirnov test (p-value = 0.3847) for overall goodness of fit is also significant. Further, the QQ-Plot (Figure 5) of actual versus simulated quantiles depicts high correlation since a straight line is obtained through a majority of points.

Finally, applying the Frank gamma model to generate predictions (Figure 6), shows a likelihood of increased future rainfall and temperature amounts.

Fig. 4. Simulated distribution values

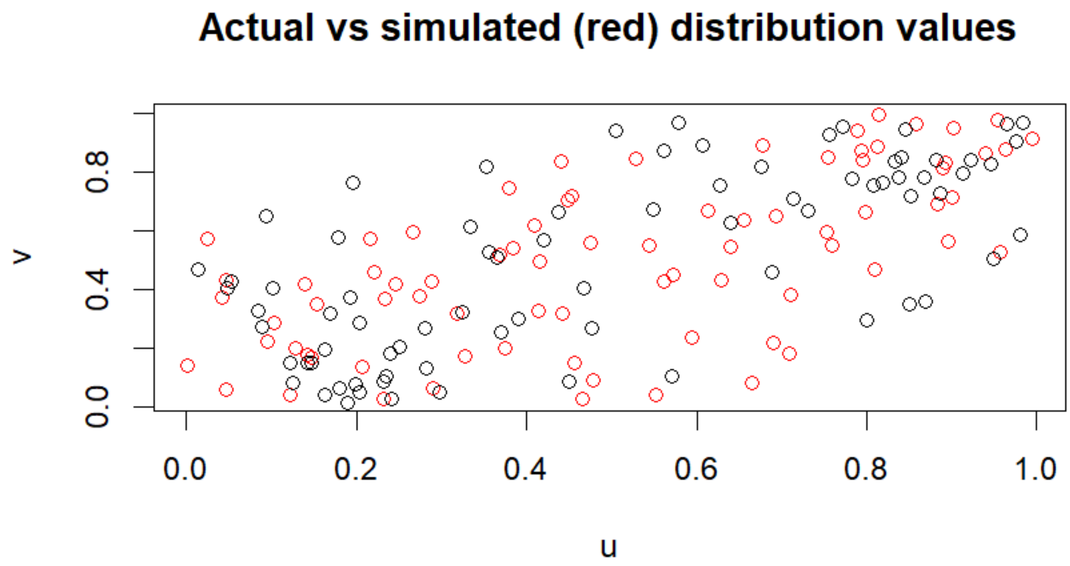


Fig. 5. Q-Q Plots

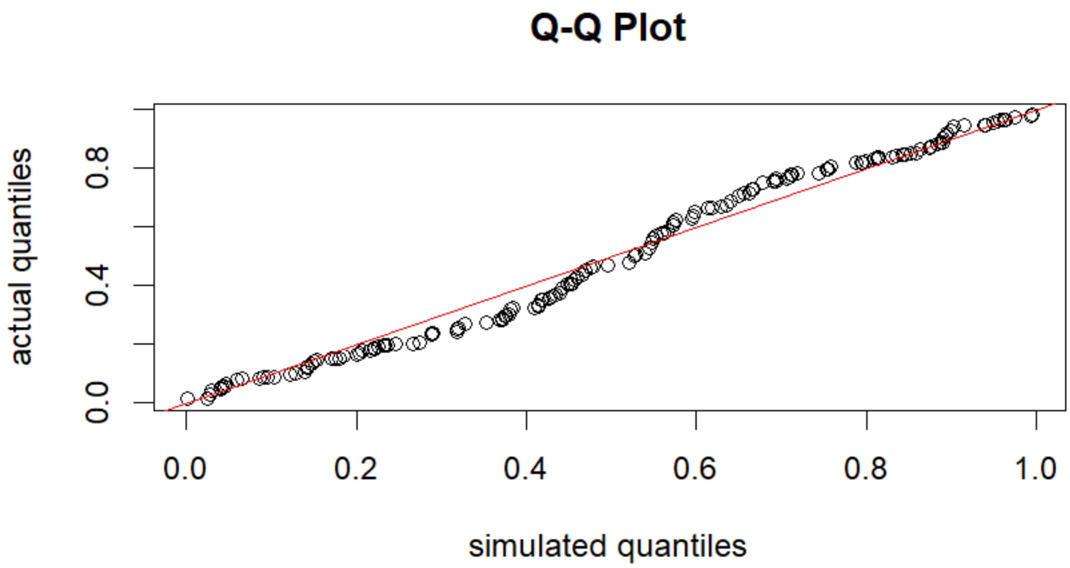
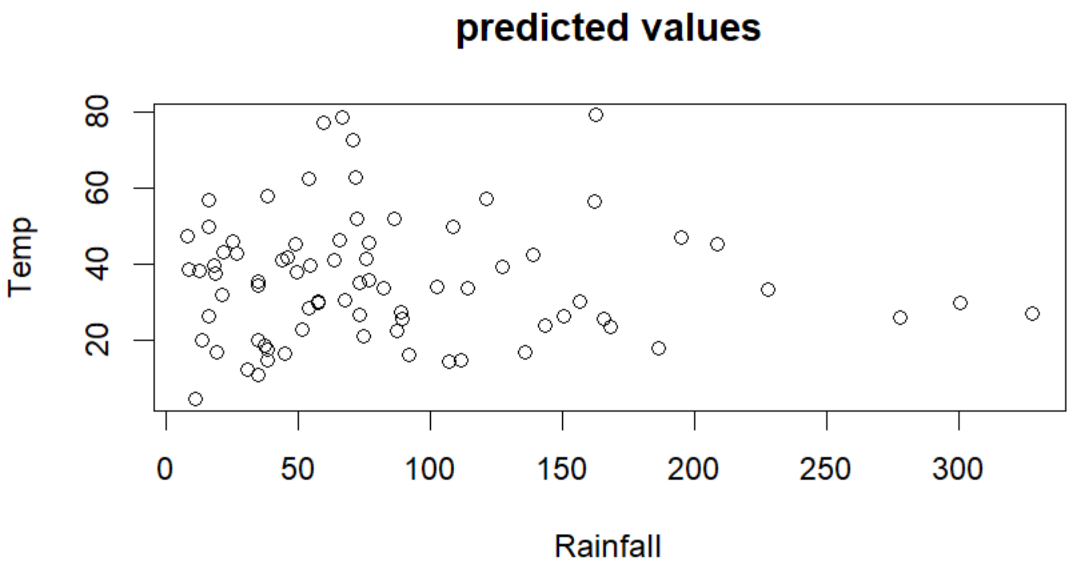


Fig. 6. Predicted future values



4. Conclusion and recommendation

In the present study, our aim is to apply the Archimedean copula approach in climate data interdependence modelling to check appropriateness of the Clayton, Gumbel and Frank copula in explaining effects of association. These approaches are applied in a secondary dataset collected from the world bank that contains Kenya rainfall and temperature data. The results indicates that the Frank copula is the most appropriate model to account for the positive association effects compared to the Gumbel or Clayton copulas. The Frank copula is therefore recommended to account for association effects that may be present in the data when modelling climate change interdependence risks in Kenya. This will ensure accurate forecasting of the weather patterns and better disaster management. Assuming independence leads to an underestimation of the extreme observation points (x). The simulated forecast values indicates the likelihood of increased future rainfall and temperature values. This information is useful both to the farmers to optimize their yields and the government for better risk mitigation measures.

Future research involves application of vine copulas and other machine learning algorithms to three or four dimensional climate data for better risk assessment of climate change probabilities. We hope to explore this area in our future research.

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4.1. Appendix

Appendix 1

Let X be gamma distributed with shape parameter p and scale parameter φ. The PDF is given by

$$f(x) = \frac{\varphi^p x^{p-1} \exp(-\varphi x)}{\Gamma(p)}; x > 0, p > 0, \varphi > 0. \tag{12}$$

The identifiable Laplace transform is given by

$$L(s) = (1 + s\sigma^2)^{-1/\sigma^2}.$$

From the definition of Archimedean copula we have that

$$C(u_1, u_2) = L[L^{-1}(u_1) + L^{-1}(u_2)], \tag{13}$$

where the generator function $\Phi(s) = L(s)$ and $\Phi^{-1}(s) = L^{-1}(s)$ is the inverse function.
Applying the Laplace transform of a gamma distribution Equation (13) becomes

$$C(u_1, u_2) = (1 + [L^{-1}(u_1) + L^{-1}(u_2)]\sigma^2)^{-1/\sigma^2}. \tag{14}$$

Substituting the inverse functions $L^{-1}(u_1) = \frac{u_1^{-\sigma^2} - 1}{\sigma^2}$ and $L^{-1}(u_2) = \frac{u_2^{-\sigma^2} - 1}{\sigma^2}$ in Equation (14) leads to

$$C(u_1, u_2) = (u_1^{-\sigma^2} + u_2^{-\sigma^2} - 1)^{-1/\sigma^2}.$$

Letting $\sigma^2 = \eta$ the definition of the Clayton copula dependence structure is obtained as

$$C(u_1, u_2) = (u_1^{-\eta} + u_2^{-\eta} - 1)^{-1/\eta}. \tag{15}$$

Appendix 2

The identifiable PDF for the positive stable is given by (Hougaard (2000))

$$f(x) = -\frac{1}{\pi x} \sum_{n=1}^{\infty} \frac{\Gamma(nr + 1)}{n!} (-x^{-\theta})^n \sin(rn\pi); x > 0, 0 < \theta \leq 1.$$

The Laplace transform is a special case of the Power Variance Family (θ, k, η) (Onchere (2022)) Laplace given by

$$L_x(s) = \exp\{-\frac{k}{\theta}s^\theta\}. \tag{16}$$

For identifiability reasons we let $k = \theta$.

$$L_x(s) = \exp\left(-s^\theta\right), 0 < \theta \leq 1. \tag{17}$$

Applying the Laplace transform of a positive stable distribution allowing for identifiability i.e. $L(s) = \exp\left(-s^\theta\right)$ Equation (13) becomes

$$C(u_1, u_2) = \exp\left(-[L^{-1}(u_1) + L^{-1}(u_2)]^\theta\right). \tag{18}$$

Substituting the inverse functions $L^{-1}(u_1) = [-\ln u_1]^{1/\theta}$ and $L^{-1}(u_2) = [-\ln u_2]^{1/\theta}$ in Equation (18) leads to

$$C(u_1, u_2) = \exp\left(-[[-\ln u_1]^{1/\theta} + [-\ln u_2]^{1/\theta}]^\theta\right).$$

Letting $\frac{1}{\theta} = \eta$ the definition of the Gumbel copula dependence structure is obtained as

$$C(u_1, u_2) = \exp\{-[(-\ln u_1)^\eta + (-\ln u_2)^\eta]^{1/\eta}\}. \tag{19}$$

Appendix 3

Fig. 7. R-Codes for fitting the Archimedean copulas

```
clayton_object<-claytonCopula(dim=2)
fitted_clayton_ml<-fitCopula(clayton_object,ttt[1:78,],method="ml")
fitted_clayton_ml
Call: fitCopula(clayton_object, data = ttt[1:78, ], ... = pairlist(method = "ml"))
Fit based on "maximum likelihood" and 78 2-dimensional observations.
Copula: claytonCopula
alpha
1.614
The maximized loglikelihood is 9.081
Optimization converged

gumbel_object<-gumbelCopula(dim=2)
fitted_gumbel_ml<-fitCopula(gumbel_object,ttt[1:78,],method="ml")
fitted_gumbel_ml
Call: fitCopula(gumbel_object, data = ttt[1:78, ], ... = pairlist(method = "ml"))
Fit based on "maximum likelihood" and 78 2-dimensional observations.
Copula: gumbelCopula
alpha
1.723
The maximized loglikelihood is 19.57
Optimization converged

frank_object<-frankCopula(dim=2)
fitted_frank_ml<-fitCopula(frank_object,ttt[1:78,],method="ml")
fitted_frank_ml
Call: fitCopula(frank_object, data = ttt[1:78, ], ... = pairlist(method = "ml"))
Fit based on "maximum likelihood" and 78 2-dimensional observations.
Copula: frankCopula
alpha
4.778
The maximized loglikelihood is 21.46
Optimization converged
```

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