



Inverted expansions of upper tails of quantile function from an extension from the the tail function (II)

Gorgui Gning ^(1,4) **M. Diallo** ^(2,4), **Gane Samb Lo** ^(3,4)

⁽¹⁾ LERSTAD, Gaston Berger University, Saint-Louis, Sénégal.

Email: gning.gorguil@ugb.edu.sn

⁽²⁾ Université des Sciences Sociale et de Gestion de Bamako (USSGB)

Faculté des Sciences Économiques et de Gestion (FSEG)

Email: moudiallo1@gmail.com.

⁽³⁾ 1178 Evanston Drive NW, T3P 0J9, Calgary, Alberta, Canada

⁽⁴⁾ Imhotep International Mathematical (IMHO-IMC). www.imhotepsciences.org

Received on January 1, 2022; June 1, 2022, Published online on September 02, 2022

Copyright © 20YY, The Journal of Mathematical Facts and Short Papers (JMFSP) and The Statistics and Probability African Society (SPAS). All rights reserved

Abstract. (English) In Diallo *et al.* (2022), the topic on inversion of the second order expansions of the tail function to obtain second order quantile function was completely introduced and described. This paper contains a new class of second expansions of the quantile functions.

(French) Dans Diallo *et al.*, le sujet sur l'inversion de la fonction de queue pour obtenir un développements du second ordre quantile de la fonction queue a été complètement introduit et décrit. Cet article contient une nouvelle classe de développements du second ordre des fonctions quantiles.

Key words: upper second order expansion; tail function; quantile function; inverting of expansion

AMS 2010 Mathematics Subject Classification : 34E05; 62G32

Corresponding author : Gorgui Gning (gning.gorguil@ugb.edu.sn)

Moumouni Diallo: moudiallo1@gmail.com

Gane Samb Lo: ganesamblo@ganesamblo.net

1. Introduction

Let F be a cumulative distribution function (cdf) on \mathbb{R} , $F(x)$, for $x \in \ell ep(F), uep(F)$, where

$$\ell ep(F) = \inf\{x \in \mathbb{R}, F(x) > 0\} \text{ and } uep(F) = \sup\{x \in \mathbb{R}, F(x) < 0\}.$$

In Extreme Value Theory (EVT), the main studies hold on the tail function $F(x)$, when x is near to $uep(F)$, and on the quantile function Q defined par $Q(0+) = \ell eo(F)$, $Q(1-) = uep(F)$ and

$$Q(1-u) = \inf\{x \in x \in \ell ep(F), uep(F)], 1 - F(x) \geq u\}, \quad 0 < u < 1.$$

The second order expansion (SOE) of $Q(1-u)$ in $u \downarrow 0$ is extremely useful in EVT and in Asymptotics of Record values. The reference of Diallo *et al.* (2022) is enough to see how to use it in these two important areas. Fundamental references can be found in de Haan (1970) and Lo *et al.* (2018), to cite a few.

Here we focus on the strict study of the inversion of the SOE of the tail function to obtain the SOE of the quantile function, and that final SOE has very notable applications as shown in

2. General expansion methods from the survival function expansions

Theorem 1. Suppose that the cdf F has an infinite upper endpoint, i.e. $uep(F) = +\infty$ and that we have the following second order expansion of $1 - F$ as $x \rightarrow +\infty$

$$1 - F(x) = A_1 x^\beta e^{-\gamma x^\eta} \left(1 + A_2 x^{-\delta_1} + O(x^{-\delta_2}) \right), \quad (1)$$

Then, as $u \downarrow 0$,

$$F^{-1}(1-u) = \left(\frac{\log(1/u)}{\gamma} \right)^{1/\eta} \left(1 + \frac{\beta \log(\gamma^{-1/\eta}) + \beta \eta^{-1} \log \log(1/u)}{\eta \log(1/u)} + O\left(\frac{\log \log(1/u)^2}{\log(1/u)^2} \right) \right).$$

Proof of Theorem 1. Let $1 - F(x) = u$ from (1), i.e.,

$$(L1) \quad A_1 x^\beta e^{-\gamma x^\eta} \left(1 + A_2 x^{-\delta_1} + O(x^{-\delta_2}) \right) = u,$$

where the assumptions on the constants are those given in the statements of the theorem. Then, we have

$$\log(u) = \log(A_1) + \beta \log(x) - \gamma x^\eta + A_2 x^{-\delta_1} + O(x^{-2\delta_1}) \quad (L2)$$

$$\begin{aligned} &= -\gamma x^\eta \left(1 - \frac{\beta}{\gamma} \log(x) x^{-\eta} - \frac{\log(A_1)}{\gamma} x^{-\eta} \right. \\ &\quad \left. - \frac{A_2}{\gamma} x^{-(\eta+\delta_1)} + O(x^{-(2\delta_1+\eta)}) \right) \\ &= -\gamma x^\eta \left(1 - \frac{\beta \log(x)}{\gamma x^\eta} + O(x^{-\eta}) \right) \quad (L3) \end{aligned}$$

From (L3), we have

$$\log(u) = -\gamma x^\eta (1 + o(1))$$

and then

$$\log \log(1/u) = \log(\gamma) + \eta \log(x) + o(1)$$

and next

$$(L4) \quad \log(x) = \frac{1}{\eta} \log \log(1/u) (1 + o(1))$$

(L3) also leads to

$$(L5) \quad x^\eta = \frac{\log(1/u)}{\gamma} \left(1 + \frac{\beta \log(x)}{\gamma x^\eta} + O\left(\frac{\log(x)^2}{x^{2\eta}}\right) \right)$$

and hence

$$(L6) \quad x = \left(\frac{\log(1/u)}{\gamma} \right)^{1/\eta} \left(1 - \frac{\beta \log(x)}{\eta \gamma x^\eta} + O\left(\frac{\log(x)^2}{x^{2\eta}}\right) \right),$$

which implies

$$(L7) \quad \log(x) = \log(\gamma^{-1/\eta}) + \frac{\log \log(1/u)}{\eta} - \frac{\beta \log(x)}{\eta \gamma x^\eta} + O\left(\frac{\log(x)^2}{x^{2\eta}}\right).$$

From (L7) next by (L5), we have

$$(L8) \quad \log(x) = \frac{1}{\eta} \log \log(1/u) (1 + o(1)) \quad \text{and} \quad x^\eta = \frac{\log(1/u)}{\gamma} (1 + o(1)),$$

and, by plugging this in (L7), we get

$$(L7a) \quad \log(x) = \log(\gamma^{-1/\eta}) + \frac{\log \log(1/u)}{\eta} + O\left(\frac{\log \log(1/u)}{\log(1/u)}\right).$$

while (L6) becomes

$$(L6a) \quad x = \left(\frac{\log(1/u)}{\gamma}\right)^{1/\eta} \left(1 + \frac{\beta \log(x)}{\eta \gamma x^\eta} + O\left(\frac{\log \log(1/u)^2}{\log(1/u)^2}\right)\right).$$

Finally, by plugging (L7a) into (L6a), we get

$$\begin{aligned} x &= \left(\frac{\log(1/u)}{\gamma}\right)^{1/\eta} \left(1 + \frac{\beta \log(\gamma^{-1/\eta}) + \beta \eta^{-1} \log \log(1/u)}{\eta \gamma \left(\gamma^{-1} \log(1/u) \left(1 + O(\log(x)x^{-\eta})\right)\right)} + O\left(\frac{\log \log(1/u)^2}{\log(1/u)^2}\right)\right) \\ &= \left(\frac{\log(1/u)}{\gamma}\right)^{1/\eta} \left(1 + \frac{\beta \log(\gamma^{-1/\eta}) + \beta \eta^{-1} \log \log(1/u)}{\eta \log(1/u)} + \frac{\beta \log(\gamma^{-1/\eta}) + \beta \eta^{-1} \log \log(1/u)}{\eta \log(1/u)} O(\log(x)x^{-\eta}) + O\left(\frac{\log \log(1/u)^2}{\log(1/u)^2}\right)\right), \end{aligned}$$

Since, by (L8), we have

$$\frac{\beta \log(\gamma^{-1/\eta}) + \beta \eta^{-1} \log \log(1/u)}{\eta \log(1/u)} O(\log(x)x^{-\eta}) = O\left(\frac{\log \log(1/u)^2}{\log(1/u)^2}\right),$$

We conclude that

$$F^{-1}(1-u) = \left(\frac{\log(1/u)}{\gamma}\right)^{1/\eta} \left(1 + \frac{\beta \log(\gamma^{-1/\eta}) + \beta \eta^{-1} \log \log(1/u)}{\eta \log(1/u)} + O\left(\frac{\log \log(1/u)^2}{\log(1/u)^2}\right)\right) \quad (2)$$

■

3. An interesting application: Gaussian cdf

Let us apply the result to one of the most important law in Distribution Studies. Let F be the cdf of a standard real-valued random variable:

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt.$$

Put $A_1 = 1/\sqrt{2\pi}$. We have for $x > 0$,

$$\begin{aligned} A_1^{-1}(1 - F(x)) &= \int_x^{+\infty} t^{-1} d(-e^{-t^2/2}) \\ &= \left[-t^{-1}e^{-t^2/2}\right]_{t=x}^{t=+\infty} - \int_x^{+\infty} t^{-2} e^{-t^2/2} dt \quad (L2) \\ &= x^{-1}e^{-x^2/2} - \int_x^{+\infty} t^{-2} te^{-t^2/2} dt \quad (L3) \\ &= x^{-1}e^{-x^2/2} - x^{-2}e^{-x^2/2} + \int_x^{+\infty} t^{-3}e^{-t^2/2} dt \quad (L4) \\ &= x^{-1}e^{-x^2/2} - x^{-2}e^{-x^2/2} + x^{-3}e^{-x^2/2} \\ &\quad - \int_x^{+\infty} t^{-4}e^{-t^2/2} dt \quad (L4) \\ &\leq x^{-1}e^{-x^2/2} - x^{-2}e^{-x^2/2} + x^{-3}e^{-x^2/2}. \end{aligned}$$

So, we have the following SOE

$$1 - F(x) = A_1 x^\beta e^{-\gamma x^\eta} (1 + A_2 x^{-\delta_1} + O(x^{-\delta_2})), \text{ as } x \rightarrow +\infty,$$

with $A_1 = 1/\sqrt{2\pi}$, $\beta = -1$, $\eta = 2$, $\gamma = 1/2$, $A_2 = 1$, $\delta_1 = 1$ and $\delta_2 = 2$. By Theorem ??, we get the SOE (??) for the quantile function as follows:

$$\begin{aligned} Q(1 - u) &= (2\log(1/u))^{1/2} \left(1 - \frac{\log \log(1/u) + \log 2}{4\log(1/u)}\right) \\ &\quad + O\left(\left(\frac{\log \log(1/u)}{\log(1/u)}\right)^2\right). \end{aligned} \tag{3}$$

4. Conclusion

We hope that result of this paper will be very useful for studies involving the tails or survival functions of cdf's. In particular, it should be of certain relevance in Extreme value theory and records theory.

References

- Diallo M., Ngom M. and Lo G.S.(2022). Inverting expansion of upper tails of quantile function from an extension from the the tail function (I). Journal of Mathematical facts and Short Papers (jmfsp). doi : <https://dx.doi.org/10.16929/jmfsp/2022.81.1>.
- de Haan, L. (1970). *On regular variation and its application to the weak convergence of sample extremes*. Mathematical Centre Tracts, **32**, Amsterdam. (MR0286156)
- Lo G.S., K. T. A. Ngom M. and Diallo M.(2018). Weak Convergence (IIA) - Functional and Random Aspects of the Univariate Extreme Value Theory. Arxiv : 1810.01625