



Inverting expansion of upper tails of quantile function from an extension from the the tail function (I)

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Abstract. Second order expansions of the quantile function $Q(1 - u)$ in the neighborhood of zero in u , are key tools for addressing the extreme value domain, the estimation of the extremal index, the asymptotic behavior of the record value, and some other extremal patterns of univariate random variable. That expansion is also a powerful tools in finding rates of convergence. In many situations, the expansion of the tail function $1 - F(x)$ is usually easier to get. From there, the expansion of the quantile function is derived from that of the tail function. Fortunately, we can find whole classes of tail expansions that are inverting to the quantile function is a unique ways. In this paper, we provide two classes of inverting expansion from the tail to the quantile. These results allow researchers to proceed without redoing heavy and complicated computations.

Key words: upper second order expansion; tail function; quantile function; inverting of expansion

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Résumé (Abstract in French) Le développement à l'ordre de la fonction quantile $Q(1-u)$ au voisinage de zéro en u , est un outil clé pour traiter établir domaine des valeurs extrêmes, l'estimation de la valeur de l'indice extremal, le comportement asymptotique des upper records et d'autres modèles extremaux de variables aleatoires sur la droite réelle. Ce développement est également un outil puissant pour trouver des vitesses de convergence. Dans de nombreuses situations, le Le développement de la fonction de queue $1-F(x)$ est généralement plus facile à obtenir. De là, Le développement de la fonction quantile est dérivée de celle de la fonction de queue. Heureusement, nous pouvons trouver des classes entières de Le développement de queues qui s'inversent à la fonction quantile en une façon unique. Dans cet article, nous fournissons deux classes Le développements inverse de la queue au quantile. Ces résultats permettent aux chercheurs de procéder sans refaire des calculs lourds et compliqués.

1. Introduction

Let F be a cumulative distribution function (cdf) on \mathbb{R} , $F(x)$, for $x \in \ell ep(F)$, $uep(F)$, where

$$\ell U(F) = \inf\{x \in \mathbb{R}, F(x) > 0\} \text{ and } uep(F)(F) = \sup\{x \in \mathbb{R}, F(x) < 0\}.$$

In Extreme Value Theory (EVT), the main studies hold on the tail function $F(x)$, when x is near to $uep(F)$, and on the quantile function Q defined par $Q(0+) = \ell U(F)$, $Q(1-) = uep(F)$ and

$$Q(1-u) = \inf\{x \in x \in \ell ep(F), uep(F)], 1 - F(x) \geq u\}, \quad 0 < u < 1.$$

The second order expansion (SOE) of $Q(1-u)$ in $u \downarrow 0$ is extremely useful in EVT and in Asymptotics of Record values. The reference of [Diallo et al. \(2021\)](#) is enough to see how to use it in these two important areas. Fundamental references can be found in [de Haan \(1970\)](#) and [?](#), to cite a few.

Here we focus on the strict study of the inversion of the SOE of the tail function to obtain the SOE of the quantile function, and that final SOE has very notable applications as shown in

2. General expansion methods from the survival function expansions

Theorem 1. Suppose that the cdf F has an infinite upper endpoint, ie $uep(F) = +\infty$ and that we have the following second order expansion of $1-F$ as $x \rightarrow +\infty$

$$1 - F(x) = A_1 x^{-\alpha_1} + A_2 x^{-\alpha_2} + O(x^{-\alpha_3}),$$

with $0 < \alpha_1 < \alpha_2 < \alpha_3$.

Then the upper quantile function $F^{-1}(1-u)$ is expanded as follows as $u \rightarrow 0$:

$$F^{-1}(1-u) = C_1 u^{-\gamma} (1 + C_2 u^\alpha + O(u^\beta)), \quad u \in]0, 1[$$

with

$$\begin{aligned} \gamma &= 1/\alpha_1, \quad C_1 = A_1^{1/\alpha_1}, \quad C_2 = A_2 A_1^{-\alpha_2/\alpha_1} / \alpha_1, \\ \alpha &= \frac{\alpha_2 - \alpha_1}{\alpha_1}, \quad \beta = \min(2(\alpha_2 - \alpha_1), \alpha_3 - \alpha_1) / \alpha_1. \end{aligned}$$

Theorem 2. Let F be a cdf with finite upper endpoint ie $uep(F) = x_0 < \infty$. Suppose that $F(x)$ has the following expansion at x_0 :

$$1 - F(x) = A_1 (x_0 - x)^{\alpha_1} + A_2 (x_0 - x)^{\alpha_2} + O((x_0 - x)^{\alpha_3})$$

with $0 < \alpha_1 < \alpha_2 < \alpha_3$.

Then the upper quantile is expanded as follows

$$x_0 - F^{-1}(1-u) = C_1 u^{-\gamma} (1 + C_2 u^\alpha + O(u^\beta))$$

with

$$\begin{aligned} \gamma &= -1/\alpha_1, \quad C_1 = A_1^{-1/\alpha_1}, \quad C_2 = A_2 A_1^{\alpha_2/\alpha_1} / \alpha_1, \\ \alpha &= (\alpha_2 - \alpha_1) / \alpha_1, \quad \beta = \min(2(\alpha_2 - \alpha_1), \alpha_3 - \alpha_1) / \alpha_1. \end{aligned}$$

Proof of Theorem 1. Let $u \in]0, 1[$ and $x \in \mathbb{R}$, as $x \rightarrow uep(F)$,

$$u = 1 - F(x) = A_1 x^{-\alpha_1} + A_2 x^{-\alpha_2} + O(x^{-\alpha_3}),$$

with $0 < \alpha_1 < \alpha_2 < \alpha_3$. We have

$$u = A_1 x^{-\alpha_1} \left(1 + A_2' x^{-\beta_2} + O(x^{-\beta_3}) \right), \tag{1}$$

with

$$A_2' = A_2 A_1^{-1}, \quad \beta_i = \alpha_i - \alpha_1, \quad (i = 2, 3), \quad \beta_2 < \beta_3.$$

So from (1)

$$x^{\alpha_1} = A_1 u^{-1} (1 + o(1)) \tag{2}$$

and from that, we get

$$x = A_1^{1/\alpha_1} u^{-1/\alpha_1} (1 + o(1))$$

and next

$$x = A_1^{1/\alpha_1} u^{-1/\alpha_1} \left(1 + O\left(u^{\beta_2/\alpha_1}\right)\right). \quad (3)$$

By combining (1) and (3), we have:

$$\begin{aligned} x^{\alpha_1} &= A_1 u^{-1} \left[1 + A_2' (A_1^{1/\alpha_1})^{-\beta_2} u^{\beta_2/\alpha_1} \left(1 + O\left(u^{\beta_2/\alpha_1}\right) + O\left(u^{\beta_3/\alpha_1}\right)\right)\right] \\ &= A_1 u^{-1} \left[1 + A_2 A_1^{-\alpha_2/\alpha_1} u^{\beta_2/\alpha_1} + O\left(u^{2\beta_2/\alpha_1}\right) + O\left(u^{\beta_3/\alpha_1}\right)\right] \\ &= A_1 u^{-1} \left[1 + A_2 A_1^{-\alpha_2/\alpha_1} u^{\beta_2/\alpha_1} + O\left(u^{2\beta_2 \wedge \beta_3/\alpha_1}\right)\right] \end{aligned}$$

By taking both members of the equality, we obtain

$$x = A_1^{1/\alpha_1} u^{-1/\alpha_1} \left[1 + A_2 A_1^{-\alpha_2/\alpha_1} u^{\beta_2/\alpha_1} + O\left(u^{2\beta_2 \wedge \beta_3/\alpha_1}\right)\right]$$

We conclude

$$F^{-1}(1 - u) = C_1 u^{-\gamma} (1 + C_2 u^\alpha + O(u^\beta))$$

with

$$\begin{aligned} C_1 &= A_1^{1/\alpha_1}, C_2 = A_2 A_1^{\alpha_2/\alpha_1} / \alpha_1, \alpha = (\alpha_2 - \alpha_1) / \alpha_1 \\ \beta &= ([2(\alpha_2 - \alpha_1)] \wedge [\alpha_3 - \alpha_2]) / \alpha_1, \gamma = 1/\alpha_1. \end{aligned}$$

Proof of Theorem 2. We have :

$$1 - F(x) = A_1 (x_0 - x)^{\alpha_1} + A_2 (x_0 - x)^{\alpha_2} + O((x_0 - x)^{\alpha_3}).$$

with $0 < \alpha_1 < \alpha_2 < \alpha_3$. Let us put $1 - F(x) = u$. We have

$$\begin{aligned} u &= A_1 (x_0 - x)^{\alpha_1} + A_2 (x_0 - x)^{\alpha_2} + O((x_0 - x)^{\alpha_3}) \\ u &= A_1 (x_0 - x)^{\alpha_1} \left[1 + \frac{A_2}{A_1} (x_0 - x)^{\alpha_2 - \alpha_1} + O\left((x_0 - x)^{\alpha_3 - \alpha_1}\right)\right] \\ (x_0 - x)^{\alpha_1} &= A_1 u^{-1} \left[1 + \frac{A_2}{A_1} (x_0 - x)^{\alpha_2 - \alpha_1} + O\left((x_0 - x)^{\alpha_3 - \alpha_1}\right)\right] \end{aligned}$$

Then

$$x_0 - x = A_1^{-1/\alpha_1} u^{1/\alpha_1} [1 + v]^{1/\alpha_1}$$

with

$$v = \frac{A_2}{A_1} (x_0 - x)^{\alpha_2 - \alpha_1} + O\left((x_0 - x)^{\alpha_3 - \alpha_1}\right) \rightarrow 0 \text{ as } x \rightarrow \text{uep}(F) = x_0$$

Then we have

$$(1 + v)^{1/\alpha_1} = 1 + \frac{1}{\alpha_1} v + O(v^2)$$

so

$$x_0 - x = A_1^{-1/\alpha_1} u^{1/\alpha_1} \left[1 + \frac{A_2}{A_1} (x_0 - x)^{\alpha_2 - \alpha_1} + O\left((x_0 - x)^{\alpha_3 - \alpha_1}\right) + O\left((x_0 - x)^{2(\alpha_3 - \alpha_1)}\right) \right]$$

Then

$$x_0 - x = A_1^{-1/\alpha_1} u^{1/\alpha_1} (1 + o(1))$$

But

$$\begin{aligned} O(x_0 - x) &= O\left(u^{\frac{1}{\alpha_1}}\right) \\ O\left((x_0 - x)^{\alpha_3 - \alpha_1}\right) &= O\left(u^{\frac{\alpha_3 - \alpha_1}{\alpha_1}}\right) \\ O\left((x_0 - x)^{2(\alpha_3 - \alpha_1)}\right) &= O\left(u^{\frac{2(\alpha_3 - \alpha_1)}{\alpha_1}}\right) \end{aligned}$$

Hence

$$x_0 - x = \left[1 + \frac{A_2}{\alpha_1} A_1^{-\alpha_2/\alpha_1} u^{\frac{\alpha_2 - \alpha_1}{\alpha_1}} + O\left(u^{\frac{\alpha_3 - \alpha_1}{\alpha_1}}\right) + O\left(u^{\frac{2(\alpha_3 - \alpha_1)}{\alpha_1}}\right) \right]$$

Then

$$x_0 - F^{-1}(1 - u) = C_1 u^{-\gamma} (1 + C_2 u^\alpha + O(u^\beta))$$

with

$$C_1 = A_1^{-1/\alpha_1}, \gamma = -1/\alpha_1, \alpha = \frac{\alpha_2 - \alpha_1}{\alpha_1}$$
$$C_2 = A_2 A_1^{-\frac{\alpha_2}{\alpha_1}} / \alpha_1, \beta = \frac{\min(2(\alpha_2 - \alpha_1), (\alpha_3 - \alpha_1))}{\alpha_1}$$

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