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A simple proof for the general C_r inequality for r between zero and one

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Abstract. We prove the following generalized C_r formula

$$\left(\sum_{i=1}^{p} x_i\right)^r \leq \sum_{i=1}^{p} x_i^r.$$

for $p > 1$, for p positive real numbers x_i, $i \in \{1, \ldots, p\}$ and $0 < r < 1$. For $r \geq 1$, the convexity of the function $0 < x \mapsto x^r$ is enough to quickly prove the classical C_r which is

$$\left(\sum_{i=1}^{p} x_i\right)^r \leq p^{r-1} \sum_{i=1}^{p} x_i^r.$$

However, for $0 < r < 1$, we do not have the convexity. Although this result may be already known, we think that our very simple proof should shared with the community.

Key words: inequality for positive power of sum of positive numbers
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Résumé (Abstract in French) Nous prouvons la formule généralisée C_r suivante

$$\left(\sum_{i=1}^{p} x_i\right)^r \leq \sum_{i=1}^{p} x_i^r.$$

pour $p > 1$, pour p nombres réels positifs x_i, $i \in \{1, \ldots, p\}$ et $0 < r < 1$. Pour $r \geq 1$, la convexité de la fonction $0 < x \mapsto x^r$ est suffisante pour prouver rapidement la formule C_r classique qui est

$$\left(\sum_{i=1}^{p} x_i\right)^r \leq p^{r-1} \sum_{i=1}^{p} x_i^r.$$

Cependant, pour $0 < r < 1$, nous n'avons pas la convexité. Bien que ce résultat soit peut-être déjà connu, nous pensons que notre démonstration très simple devrait être partagée avec la communauté.

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1. Introduction

Let $p > 1$, x_i, $i \in \{1, \ldots, p\}$ p positive numbers and $r \geq 1$. Let us recall that the mapping $0 < x \mapsto x^r$ is convex. We can use that convexity to quickly get a bound of

$$\left(\sum_{i=1}^{p} x_i\right)^r$$

Indeed, we have

$$\begin{aligned}
\left(\sum_{i=1}^{p} x_i\right)^r &= \left(p \sum_{i=1}^{p} \frac{1}{p} x_i\right)^r \\
&= p^r \left(\sum_{i=1}^{p} \frac{1}{p} x_i\right)^r
\end{aligned}$$

Now we can apply the convexity to the convex combination

$$\left(\sum_{i=1}^{p} \frac{1}{p} x_i\right)^r$$

to get

$$\left(\sum_{i=1}^{p} x_i\right)^r \leq p^r \sum_{i=1}^{p} \frac{1}{p} x_i^r$$

This leads to the C_r inequality

$$\left(\sum_{i=1}^{p} x_i\right)^r \leq p^{r-1} \sum_{i=1}^{p} x_i^r$$

It happens that in some areas of Mathematics where we need a similar inequality for $r \in]0,1[$. For example, this is the case in the proof of the Kolmogorov continuity theorem (see Revusz and Yor).

Preparing on his book on stochastic process, Prof Lo told me about that kind of inequality that Revusz and Yor cited without proof. I tried and found this proof. We think that this proof should be shared for at least a pedagogical reason and a vulgarization need. We present it it the next section.

2. Result and proof

Theorem 1. *For $r \in]0,1[$, we have*

$$\left(\sum_{i=1}^{p} x_i\right)^r \leq \sum_{i=1}^{p} x_i^r.$$

To prove this, we need the following result

Lemma 1. *The mapping $f : t \longrightarrow t^r$, $r \in]0,1[$ defined on $\mathbb{R}_+$ is nondecreasing such as $f(0) = 0$ and for all $(x,y) \in \mathbb{R}_+^2$,*

$$f(x+y) \leq f(x) + f(y)$$

Proof of Lemma 1 : Let $a \in [0,1]$ such as $x = (x+y)a$. We have $y = (x+y)(1-a)$. For $r \in]0,1[$, we have

$$a \leq a^r$$

and

$$(1-a) \leq (1-a)^r,$$

so that

$$\begin{aligned}(x+y)^r = (a + (1-a))(x+y)^r &\leq (a^r + (1-a)^r)(x+y)^r \\ &= (a(x+y))^r + ((1-a)(x+y))^r \\ &= x^r + y^r.\end{aligned}$$

Proof of Theorem 1 : We are going to proceed by recurrence on p. For $p = 1$, we have

$$x_1^r \leq x_1^r.$$

For $p = 2$, we have, by the Lemma 1,

$$(x_1 + x_2)^r \leq x_1^r + x_2^r.$$

Let us assume the relation is true to order p and show to order $p + 1$. We have

$$\begin{aligned}
\left(\sum_{i=1}^{p+1} x_i\right)^r &= \left(\sum_{i=1}^{p} x_i + x_{p+1}\right)^r \\
&\leq \left(\sum_{i=1}^{p} x_i\right)^r + x_{p+1}^r \\
&\leq \sum_{i=1}^{p} x_i^r + x_{p+1}^r \\
&= \sum_{i=1}^{p+1} x_i^r.
\end{aligned}$$

The proof is complete by induction.

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References

Revusz and Jacod ().