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The fundamental property for slowly varying functions at zero

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Abstract. A function $(0,1) \ni u \mapsto s(u) \in \mathbb{R}_+$ is slowly varying at zero if and only if: for $\lambda > 0$, $s(\lambda u)/s(u)$ converges to 1 (one) as u goes to zero. The fundamental property says that, for any positive random variables $A \in (0,1)$ and $B \in (0,1)$, the limit above can be generalized as: let $(u_n)_{n\geq 1} \subset (min(A,B),\ max(A,B))$ such that $u_n \rightarrow_{\mathbb{P}} 0$ as $n \rightarrow +\infty$ then

$$\left(\sup_{u \in [min(A,B), max(A,B)],} \left| \frac{s(u)}{s(u_n}) - 1 \right| \right)^{\star} \rightarrow_{\mathbb{P}} 0.$$

where $A^{\star}$ stands as the maximal measurable cover of the set $A \subset \Omega$. That property is extended to families of bounds eventually non-measurable (see page 70 for the full abstract).

Key words: regularly varying function; slowly varying function; Karamata's representation; etc.
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Full Abstract A function $(0,1) \ni u \mapsto s(u) \in \mathbb{R}_+$ is slowly varying at zero if and only if: for $\lambda > 0$, $s(\lambda u)/s(u)$ converges to 1 (one) as u goes to zero. The fundamental property says that, for any positive random variables $A \in (0,1)$ and $B \in (0,1)$, the limit above can be generalized as: let $(u_n)_{n\geq 1} \subset (min(A,B),\ max(A,B))$ such that $u_n \rightarrow_{\mathbb{P}} 0$ as $n \rightarrow +\infty$ then

$$\left(\sup_{u\in[min(A,B),max(A,B)],} \left|\frac{s(u)}{s(u_n}) - 1\right|\right)^{\star} \rightarrow_{\mathbb{P}} 0.$$

as $n \rightarrow +\infty$, where for any mapping $(\Omega, \mathcal{A}, \mathbb{P}) \mapsto \overline{\mathbb{R}}$, $X^{\star}$ stands upper measurable cover, i.e., the infimum of all random variable greater or equal to X. In its turn, that later version can be extended for A and B replaced with families $(A(h))_{h\in D}$ and $(B(h))_{h\in D}$ in $(0,1)$ where D is an upward directed set, such that $(A(h)/B(h))$ and $(B(h)/A(h))$ are bounded in probability as h goes upwards on D. This result is the key tool that surved as a portal for the successful study of statistical properties for the Gumbel type of extremes as proposed by Lo (1986). He we give the most general version with the possibility of the non-measurability of A, B, $A(h)$ and $B(h)$.

Résumé (Abstract in French). Une fonction $(0,1) \ni u \mapsto s(u) \in \mathbb{R}_+$ est à variation lente dans le voisinage de zéro si et seulement si : pour tout $\lambda > 0$, $s(\lambda u)/s(u)$ converge vers 1 (un) lorsque u tend vers zéro. La propriété fondamentale affirme que, pour deux variables aléatoires positives $A \in (0,1)$ et $B \in (0,1)$, la limite ci-dessus peut être généralisée comme suit : soit $(u_n)_{n\geq 1} \subset (\min(A,B), \max(A,B))$ tel que $u_n \rightarrow_{\mathbb{P}} 0$ quand $n \rightarrow +\infty$, alors

$$\left(\sup_{u\in[\min(A,B),\max(A,B)]} \left|\frac{s(u)}{s(u_n)} - 1\right|\right)^{\star} \rightarrow_{\mathbb{P}} 0,$$

quand $n \rightarrow +\infty$, où pour toute application $(\Omega, \mathcal{A}, \mathbb{P}) \mapsto \overline{\mathbb{R}}$, $X^{\star}$ désigne la couverture supérieure mesurable, c'est-à-dire, la plus petite variable aléatoire supérieure ou égale à X. Un tel résultat peut être étendu lorsque A et B sont remplacés par des familles $(A(h))_{h\in D}$ et $(B(h))_{h\in D}$ dans $(0,1)$, où D est un indexant bien dirigé vers le haut, telles que $(A(h)/B(h))$ et $(B(h)/A(h))$ sont bornées en probabilité quand h parcourt D. Ce résultat est la clé pour étudier les propriétés statistiques des extrêmes de type Gumbel comme proposé par Lo (1986). Nous donnons ici sa version la plus générale avec la possibilité de non-mesurabilité de A, B, $(A(h))$ et $(B(h))$.

Presentation of the author.

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1. Introduction

Functions $(0,1) \ni u \mapsto s(u) \in \mathbb{R}_+$ and $\mathbb{R}_+ \ni x \mapsto S(x) \in \mathbb{R}_+$ at zero and at $!infty$ slowly varying at zero and at $+\infty$ only if: for $\lambda > 0$,

$$\lim_{u \rightarrow 0} \frac{s(\lambda u)}{s(u)} \rightarrow = 1, \quad as \quad u \rightarrow 0,$$

and

$$\lim_{x \rightarrow +\infty} \frac{S(\lambda x)}{S(x)} \rightarrow = 1, \quad as \quad x \rightarrow +\infty,$$

respectively. Studying slow variations at zero and at infinity are equivalent through the change of variables $x\ u = 1$. Here we are focusing on slow variations at zero.

Slowly varying functions are very important in real analysis and are close related to regularly varying functions

$$u \mapsto u^{\alpha}\ s(u)$$

where s is slowly varying. They play important roles in a great number of fields such as ... However, their power is fully taken into account in extreme value theory on $\mathbb{R}$. in the second half of the 1980's, the weak normality of largest observations of sample was a serious topic. Within the class of distribution functions in the extremal domain of attraction, the solution for distributions in the Gumbel type of attraction was surely the most difficult and was declared as a conjecture. Lo (1986) has been able to establish the fundamental property as described in the abstract that has been instrumental to the final solution of that conjecture (see Lo (1989) as reported by Csorgo *et al.* (1987)).

Years after, we have the pleasure to give this this most general form of that fundamental property. Before we begin, we have to recall the so important Karamata (1930)'s representation of slowly varying function.

Lemma 1. *A measurable function* $(0,1) \ni u \mapsto s(u) \in \mathbb{R}_+$ *is slowly varying at zero if and only if there exists a constant* $c > 0$ *and functions* $a(u)$ *and* $\ell(u)$ *such that* $(a(u), \ell(u)) \rightarrow (0,0)$ *satisfying*

$$s(u) = c(1 + a(u))\ \exp\left(\int_1^u \frac{\ell(t)}{t}\ dt\right), \quad u \in (0,1). \tag{1}$$

Important developments on regular or slowly varying functions are available in Lo (1986), Galambos (1978), de Haan (1970), Resnick (1987), Feller (1968b), Karamata (1930), etc.

Here is the result of this paper.

2. The fundamental property of slowly varying functions

We have:

Theorem 1. *Let $S(u)$ be a function $u \in (0, 1)$ that is slowly varying at zero. We have the following uniform convergence in deterministic and random versions.*

(a) Let $A(h)$ and $B(h)$ two functions of $h \in (0, +\infty[$ such that for each $h \in (0, +\infty[$, we have $0 < A(h) \leq B(h) < +\infty$ and $(A(h), B(h)) \to (0, 0)$ as $h \longrightarrow 0$. Suppose that there exist two real numbers $0 < C < D < +\infty$ such that

$$C < \lim_{h \to +\infty} \inf A(h)/B(h), \ \ \lim_{h \to +\infty} \sup B(h)/A(h) < B. \tag{2}$$

Then, we have

$$\lim_{h \to + \to} \sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| = 0.$$

(b) Let $A(h)$ and $B(h)$ two families, indexed by $h \in (0, +\infty[$, of real-valued applications defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that for each $h \in (0, +\infty[$, we have $0 < A(h) \leq B(h) < +\infty$. Suppose that there exist two families $A^*(h)$ and $B^*(h)$, indexed by $h \in (0, +\infty[$, of measurable *real-valued applications defined on $(\Omega, \mathcal{A}, \mathbb{P})$ such that for each $h \in (0, +\infty[$, $A^*(h) \leq A(h) \leq B(h) \leq B^*(h)$, and such that*

$$\lim_{h \to +\infty} \sup \lim_{\lambda \to +\infty} \inf \mathbb{P}(B^*(h)/A(h) > \lambda) = 0. \tag{3}$$

and

$$\lim_{h \to +\infty} \sup \lim_{\lambda \to +\infty} \inf \mathbb{P}(A^*(h)/B^*(h) < 1/\lambda) = 0. \tag{4}$$

We say that the family $\{B^*(h), h \in h \in (0, +\infty[\}$ is asymptotically bounded in probability against $+\infty$ and the family $\{B^*(h), h \in h \in (0, +\infty[\}$ is asymptotically bounded in probability against 0 and accordingly, we say that the family $\{B(h), h \in h \in (0, +\infty[\}$ is asymptotically bounded in outer probability against $+\infty$ and the family $\{A(h), h \in h \in (0, +\infty[\}$ is asymptotically bounded in outer probability against 0.

Then any $\eta > 0$, for any $\delta > 0$, there exists a measurable subset $\Delta(\delta)$ of such that

$$\left(\sup_{A(h) \leq u, v \leq B(h)} \left| \frac{S(u)}{S(v)} - 1 \right| > \eta \right) \subset \Delta(\delta),$$

with

$$\mathbb{P}(\Delta(\delta)) \leq \delta.$$

Consequently, if the quantities

$$\sup_{A(h)\leq u,v\leq B(h)} \left|\frac{S(u)}{S(v)} - 1\right| > \eta$$

are measurable for $h \in h \in (0, +\infty[$, *we have that*

$$\sup_{A(h)\leq u,v\leq B(h)} \left|\frac{S(u)}{S(v)} - 1\right| \to_{\mathbb{P}} as\ h \to +\infty.$$

Proof. Let us use the Kamarata Representation Theorem 1 of S : there exist a constant c and functions $a(u)$ and $b(u)$ of $u \in]0, 1]$ satisfying

$$(a(u), b(u)) \to (0, 0) \text{ as } u \to +\infty,$$

such that S is written as

$$S(u) = c(1 + a(u)) \exp(\int_u^1 \frac{b(t)}{t} dt).evt.karamata.theo \tag{5}$$

Proof of Point (a). Suppose that Condition (2) holds. Let $\varepsilon > 0$ such that $\varepsilon < 1$. Then two functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ of $\varepsilon \in]0, 1[$,

$$h_1(\varepsilon) = C^{\varepsilon} \frac{1-\varepsilon}{1-\varepsilon} \text{ and } h_2(\varepsilon) = D^{\varepsilon} \frac{1+\varepsilon}{1-\varepsilon}$$

both converge to 0 as $\varepsilon \downarrow 0$. So for any $\eta > 0$, there exist ε_0, $0 < \varepsilon < \varepsilon_0 < 1$, such that

$$1 - \eta \leq h_1(\varepsilon), h_2(\varepsilon) \leq 1 + \eta. \tag{6}$$

So, let $\eta > 0$ and let $\varepsilon_0 < 1$ such that (6) holds. Fix ε, $0 < \varepsilon < \varepsilon_0$. Now, by the assumptions on the functions b and p, there exists t_0 such that for $0 \leq t \leq t_0$

$$\max(|p(t)|, |b(t)|) \leq \varepsilon.$$

Since $B(h) \to 0$ as $h \to +\infty$, and since (2) holds, there is a value $h_0 > 0$ such that $h \geq h_0$ implies that $0 \leq B(h) \leq t_0$ and

$$C \leq A(h)/B(h) \text{ and } B(h)/A(h) \leq D.$$

Then for $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$, we have

$$\begin{aligned}
\frac{S(u)}{S(v)} &= \frac{1+p(u)}{1+p(v)} \exp\left(\int_u^v \frac{b(t)}{t} dt\right) &(7)\\
&\leq \frac{1+\varepsilon}{1-\varepsilon} \exp\left(\sup_{0\leq t\leq t_0} |b(t)| \int_u^v \frac{dt}{t}\right) &(8)\\
&\leq \frac{1+\varepsilon}{1-\varepsilon} \exp\left(\varepsilon \log\left\{\frac{\max(u,v)}{\min(u,v)}\right\}\right).
\end{aligned}$$

Thus for $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$, we have

$$\frac{S(u)}{S(v)} \leq D^{\varepsilon} \frac{1+\varepsilon}{1-\varepsilon}.$$

By using lower bounds on place of upper bounds in (8), we also have $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$,

$$\frac{S(u)}{S(v)} \geq C^{\varepsilon} \frac{1-\varepsilon}{1-\varepsilon}.$$

By putting together the previous facts, we have, for $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$

$$1 - \eta \leq \frac{S(u)}{S(v)} \leq 1 + \eta. \tag{9}$$

This implies that for for any $\eta > 0$, we have found h_0 such that for $h \geq h_0$ and $(u, v) \in [A(h), B(h)]^2$, we have

$$\sup_{A(h)\leq u,v\leq B(h)} \left|\frac{S(u)}{S(v)} - 1\right| \leq \eta.$$

Thus

$$\lim_{h\to+\infty} \sup_{A\leq u,v\leq B} \left|\frac{S(u)}{S(v)} - 1\right| = \lim_{h\to+\infty} \sup_{A\leq u,v\leq B} \left|\frac{S(u)}{S(v)} - 1\right| = 0.$$

Proof of Point (b). Suppose that the conditions of this point hold. For any $\delta > 0$, there exist a real number $h_1 > 0$, and a number $\lambda > 0$ ($\lambda = \lambda(\delta)$ and a real number $h_1 = h_1(\delta)$ both depend *on* δ) such that

$$\mathbb{P}(\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda) \geq 1 - \delta/2.$$

If the latter property holds for a number $\lambda > 0$, it also holds for any greater number. So, we may and do choose $\lambda > 1$. Put $C = \lambda^{-1}$ and $D = \lambda$. From here, we follow partially use the proof of Point (a). Let $\eta > 0$ and consider ε_0, $0 < \varepsilon_0 < 1$ such that for any $0 < \varepsilon < \varepsilon_0 < 1$, whe have

$$1 - \eta \leq h_1(\varepsilon), h_2(\varepsilon) \leq 1 + \eta. \tag{10}$$

And let $t_0 > 0$ such that for any $0 \leq t \leq t_0$

$$\max(|p(t)|, |b(t)|) \leq \varepsilon.$$

Since $B^*(h) \rightarrow_P 0$ as $h \rightarrow 0$, there exists a value $h_2 > 0$ such that for any $h \geq h_2$, we have

$$\mathbb{P}(B^*(h) > t_0) < \delta/2.$$

Denote $h_0 = \max(h_1, h_2)$. The conditions under which (9) was proved are satisfied on the event $(\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda) \cap (B^*(h) < t_0)$, $h \geq h_0$. Hence, we have on $(\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda) \cap (B^*(h) < t_0)$, for $h \geq h_0$

$$\sup_{A^*(h) \leq u, v \leq B^*(h)} \left| \frac{S(u)}{S(v)} - 1 \right| \leq \eta.$$

Let us denote

$$\Delta(\delta, h)^c = (\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda) \cap (B^*(h) \leq t_0).$$

We have for $h \geq h_0$,

$$\begin{aligned}\mathbb{P}(\Delta(\delta, h)) &\leq \mathbb{P}((\lambda^{-1} \leq A^*(h)/B^*(h), B^*(h)/A^*(h) \leq \lambda)^c) + \mathbb{P}(B^*(h) < t_0) \\ &\leq \delta/2 + \delta/2 = \delta.\end{aligned}$$

We also have

$$\Delta(\delta)^c \subset \left(\sup_{A^*(h) \leq u, v \leq B^*(h)} \left| \frac{S(u)}{S(v)} - 1 \right| \leq \eta \right)$$

This gives for $h \geq h_0$,

$$\left(\sup_{A(h)\leq u,v\leq B(h)}\left|\frac{S(u)}{S(v)}-1\right|>\eta\right)\subset\left(\sup_{A(h)\leq u,v\leq B^{*}(h)}\left|\frac{S(u)}{S(v)}-1\right|>\eta\right)\subset\Delta(\delta)$$

Thus for any $\eta > 0$, for any $\delta > 0$, we have found $h_0 > 0$ such that for $h \geq h_0$,

$$\left(\sup_{A(h)\leq u,v\leq B(h)}\left|\frac{S(u)}{S(v)}-1\right|>\eta\right)\subset\Delta(\delta),$$

with $\mathbb{P}(\Delta(\delta)) \leq \delta$.

The proof is complete ■

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