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An extended presentation of the rejection-method for generating a random vector using modern measure theory

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Abstract. A real random variable can be generated simply using the Rényi representation. Its computer implementation is very feasible as long as there exists an algorithm for calculating the associated quantile function. However, extending this method to higher dimensions, on $\mathbb{R}^k$, where $k > 1$, is no longer possible. On the other hand, the rejection method is more general. Indeed, it works for a real k-vector ($k \geq 1$) possessing a density with respect to the Lebesgue measure. However, it is implementable only if this density is bounded by a multiple of another density for which a generator already exists. This note is a contribution to a clearer presentation of the method within the modern framework of measure theory and probability. Our proofs serve as an alternative to the demonstrations available in Devroye (1986) and Hőrmann, Leydold, and Derflinger (2004).

Key words: data generation; reject method; measure theory and integration; probability theory
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Résumé (Abstract in French) Une variable aléatoire réelle peut être générée simplement en utilisant la représentation de Rényi. Son implémentation sur ordinateur est très possible dès qu'il existe un algorithme de calcul de la fonction des quantiles associée. Cependant l'extenstion de cette méthode en dimension supérieure, sur $\mathbb{R}^d$, $d > 1$ n'est plus possible. Par contre la methode de rejet est plus générale. En efftet, elle marche pour un k-vecteur réel ($k \geq 1$) possédant une densité par rapport à la mesure de Lebesgue. Cependant, elle n'est implémentable que si cette densité est majorée par un multiple d'une autre densité pour laquelle il existe déjà un générateur. Cette note est une contribution sur une présentation plus comprehensible de la méthode dans le cadre moderne de la théorie de la mesure et des probabilités. Nos preuves se posent en alternative aux démonstrations disponibles dans Devroye (1986) et Hőrmann, Leydold et Derflinger (2004).

1. Introduction

A random vector $X \in \mathbb{R}^d$ associated to f, a probability distribution function (pdf) with respect to the Lebesgue-measure λ_d of $\mathbb{R}^d$. We are interested in generation methods of X given f. One of the most general method is the reject method. The reject method allows to generate X at the condition there exits a density g and a scalar $\alpha \leq 1$ such that $0 \leq f \leq \alpha\ g$. This algorithm is given in Theorem below. But the numerical implementation is possible only if there exists a generator of a random vector Z associated to the *pdf* g.

The paper is organized as follows. In Section 2, we give foundational results (Theorem 1, page 58, Theorem 2, page 63) on probability laws of some special random vectors in relation with the uniformly distributed random vectors. These facts are combined in Section 2 to give the final Algorithm (RM) [page 3, 67]. In Section 4, we give concluding remarks.

2. Theoretical tools

We has this first result.

Theorem 1. *Let $d \geq 1$, $D \in \mathcal{B}(\mathbb{R}^d)$, f a non-negative and integrable function on $D \subset$, and $c \in \mathbb{R}$, $c > 0$.*

(A) Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}^d$ be a random vector supported by D, i.e

$$D =: \mathcal{V}_X = \left\{x \in \mathbb{R}^d, f(x) > 0\right\}$$

and $U : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow [0, 1]$ be a random variable uniformly distributed on $(0, 1)$, independent of X. Let

$$S_c = \{(x, u) \in (D \times \mathbb{R}), \ \ 0 \leq u \leq cf(x)\}.$$

Then, if f is the pdf of X, the pair $Z = (X,\ cUf(X))$ follows a uniform law on S_c.

(B) Let X be any random vector supported by D such that the pair

$$Z = (X,\ cUf(X))$$

is uniformly distributed on S_c, then cf is a λ_d-pdf and

$$d\mathbb{P}_X(x) = cf(x)\ 1_D(x)\ d\lambda_d(x),$$

i.e, for X has the pdf

$$f(x) = (1/\alpha)1_D(x)f(x), \ \ with \ \ \alpha = \int_D f(t)\ d\lambda_d t.$$

Before we proceed to the proof, let us define:

Definition 1. An absolutely continuous uniform law on $\mathbb{R}^d$ with support S is defined if and only if

$$0 < \lambda(S) < +\infty$$

and the related *pdf* is

$$f(x) = \frac{1}{\lambda(S)} 1_S(x), \ \ x \in \mathbb{R}.$$

we denote that uniform law on S as : $\mathcal{U}(S)$.

Remark: if $X \sim \mathcal{U}(S)$, we have for any measurable function $h : \mathbb{R}^d \to \mathbb{R}$ such that $h(X)$ is quasi-integrable, we have:

$$\mathbb{E}(h(X)) = \frac{1}{\lambda(S)} \int_S h(x)dx.$$

Proof of Theorem 1.

Proof of part (A). Let us begin to prove that $\lambda_{d+1}(S_c) = c$. By Tonelli's theorem, we have

$$\begin{aligned}
\lambda_{d+1}(S_c) &= \int_{(x,u)\in\mathbb{R}^d\times\mathbb{R}, 0\leq u\leq cf(x)} d\lambda_d(x)d\lambda(x)\\
&= \int_{x\in\mathbb{R}^d} d\lambda_d(x)\left[\int_{0\leq u\leq cf(x)} d\lambda(x)\right]\\
&= c\int_{x\in\mathbb{R}^d} f(x)d\lambda_d(x) = c,
\end{aligned}$$

since f is a *pdf* on $\mathbb{R}^d$. Next, let us denote $Z = (X, cUf(X))$. To conclude the proof, we have to prove that

$$\forall B \in \mathcal{B}(R^{d+1}),\ \ \mathbb{P}(Z \in B) = \frac{\lambda_{d+1}(B\cap S_c)}{\lambda_{S_c}}.$$

Now, by denoting $C = \{(x,u) \in D\times[0,1], (x,\ cuf(x) \in B\}$, we write

$$\begin{aligned}
\mathbb{P}(Z\in B) &= \int_{(x,u)\in D\times[0,1],(x,\ cuf(x)\in B} d\mathbb{P}_{(X,U)}(x,u)\\
&=: \int_C d(\mathbb{P}_X\otimes\mathbb{P}_U)(x,u).
\end{aligned}$$

For $x \in D$, the section of C at x is

$$\begin{aligned}
C_x &= \{u\in[0,1],\ (x, cuf(x)\in B\} = \{u\in[0,1],\ cuf(x)\in B_x\}\\
&= \{u\in[0,1],\ cuf(x)\in B_x\}\\
&= \{u\in[0,1], u\in B_x/(cf(x))\}.
\end{aligned}$$

The division in the last line is justified since $f \neq 0$ λ_d-*a.e.* on D. Hence we have

$$C_x = \frac{B_x}{cf(x)}\cap[0,1],\ \ \lambda_d\ a.e.$$

Let $x \in D$ and $u \in \{B_x/(cf(x))\} \cap [0,1]$. Then there exists $t \in B_x$ such that $u = t/(cf(x)) \in [0,1]$. Hence $0 \leq t \leq cf(x)$ and then $f \in (S_c)_x$. Then

$$B_x/(cf(x))\}\cap[0,1] \subset (B_x\cap(S_c)_x)/(cf(x))$$

and next

$$B_x/(cf(x))\} \cap [0,1] = (B_x \cap (S_c)_x)/(cf(x)).$$

Finally, by using this fact, as explained again below, we get

$$\begin{aligned}
\mathbb{P}(Z \in B) &= \int_{x\in D} \lambda_{[0,1]} \left(\frac{(B_x}{cf(x)}\right) f(x) d_{\lambda_d}(x) \\
&= \int_{x\in D} \lambda_{[0,1]} \left(\frac{(B \cap S_c)_x}{cf(x)}\right) f(x) d_{\lambda_d}(x) \\
&= \int_{x\in D} \lambda \left(\frac{(B \cap S_c)_x}{cf(x)}\right) f(x) d_{\lambda_d}(x) \\
&= \int_{x\in D} \frac{1}{cf(x)} \lambda\left((B \cap S_c)_x\right) f(x) d_{\lambda_d}(x) \\
&= \frac{1}{c} \int_{x\in D} \lambda\left((B \cap S_c)_x\right) \quad d_{\lambda_d}(x) \\
&= \frac{1}{c} \int_{x\in D} \int_{u\in B_x} 1_{S_c}(x,u) d_{\lambda_d}(x) d\lambda(u) \\
&= \frac{1}{c} \int_{(x,u)\in D\times\mathbb{R}} 1_{B\cap S_c}(x,u) d_{\lambda_{d+1}}(x,u) \\
&= \frac{\lambda_{d+1}(B \cap S_c)}{\lambda_{d+1}(S_c)}.
\end{aligned}$$

We have to make two remarks about the equalities above. From Line 1 to Line 2, We were able to drop the restriction of the Lebesgue measure on $[0,1]$ since $(B \cap S_c)/(cf(x))$ is in $[0,1]$ as remarked earlier. From Line 2 to Line 3, we use the classical result of the Lebesgue measure on $\mathbb{R}^d$: $\lambda(\alpha B) = |\alpha|^d \, \lambda(B)$ for any Borel set $B \in \mathcal{B}(\mathbb{D}^d)$. The proof of Part B is finished.

Proof of Part(B). Let X be a random vector such that $(X, cUf(X))$ be uniformly distributed on S_c. Let us show that

$$d\mathbb{P}_X(x) = cf(x)1_D(x)d\lambda_d(x).$$

Let $\Delta \in \mathcal{B}(\mathbb{R}^d)$ and $\bar{\Delta} = \Delta \times \mathbb{R}$. We have

$$\begin{aligned}
\mathbb{P}_X(\Delta) &= \mathbb{P}(X \in \Delta) \\
&= \mathbb{P}(Z \in \bar{\Delta}) \\
&= \frac{\lambda_{d+1}(\bar{\Delta} \cap S_c)}{\lambda_{d+1}(S_c)}.
\end{aligned}$$

Now, we obtain

$$\begin{aligned}\lambda_{d+1}(\bar{\Delta} \cap S_c) &= \int_{x \in \Delta \cap D, \quad 0 \leq u \leq cf(x)} d\lambda_d(x, u). \\ &= c \int_D 1_D(x) f(x) d_{\lambda_d}(x)\end{aligned}$$

We also have

$$\begin{aligned}\lambda_{d+1}(S_c) &= \int_{u \in D, \quad 0 \leq u \leq cf(x)} d_{\lambda_d}(x) d\lambda(x) \\ &= c \int_D f(x) d\lambda_d(x).\end{aligned}$$

So,

$$\mathbb{P}_X(\mathbb{R}^d) = c \int_D f(x) d\lambda_{d+1}(x) = 1.$$

Then

$$g(x) = c\ f(x)\ 1_D(x),$$

is an **ac-pdf** and

$$\mathbb{P}_X(\Delta) = \int_\Delta g(x) d\lambda_d(x).$$

We get

$$d\mathbb{P}_X(x) = c 1_D(x) f(x) dd\lambda_d(x).$$

We have a second theorem

Theorem 2. *Let X, X_2,... be a sequence of independent and identically random vectors in $\mathbb{R}^d$, $d \geq 1$ defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with support D. Let $D' \subseteq \mathbb{R}^d$ such that $p = \mathbb{P}(X \in D') > 0$. Define*

$$\tau = \inf\{j \geq 1, X_j \in D'\}.$$

Then τ is finite and

$$\forall B \in \mathcal{B}(\mathbb{R}^d), \mathbb{P}(X_\tau \in B) = \frac{\mathbb{P}(X \in B \cap A)}{p}$$

Suppose that $D \supseteq D'$. We have:

If X is uniformly distributed on D, then X_τ is uniformly distributed on D'.

Proof. Let us begin to see that τ is finite *a.e.*. Indeed, for $p = 1$, then $D \subseteq D'$ a.e, and $\tau = 1$ and

$$\mathbb{P}(X_1 \in B) = \mathbb{P}(X \in D' \cap B).$$

Let $p \in]0, 1[$. We have

$$\begin{aligned}
\mathbb{P}(\tau = +\infty) &= \mathbb{P}(\forall j \geq 1, X_j \notin D') \\
&= \mathbb{P}(X_1 \notin D')^\infty = (1-p)^\infty = 0.
\end{aligned}$$

So, τ is finite *a.e.*. Suppose that the assumptions of the theorem hold. We have, for any $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}
\mathbb{P}(X_\tau \in B) &= \sum_{j=1}^{\infty} \mathbb{P}(X_\tau \in B, \tau = j) \\
&= \sum_{j=1}^{\infty} \mathbb{P}(X_1 \notin D', ..., X_{j-1} \notin D', X_{j-1} \notin D' \cap B, \tau = j) \\
&= \sum_{j=1}^{\infty} \mathbb{P}(X_1 \notin D', ..., X_{j-1} \notin D', X_j \in D' \cap B)
\end{aligned}$$

To justify that $(\tau = j)$ might be dropped in the last line comes from that

$$(X_1 \notin D', ..., X_{j-1} \notin D', X_j \in D' \cap B) \subset (\tau = j).$$

Now, we have

$$\begin{aligned}\mathbb{P}(X_\tau \in B) &= \sum_{j=1}^{\infty} \mathbb{P}(X \notin D')^{j-1} \mathbb{P}(X \in D' \cap B) \\ &= \mathbb{P}(X \in D' \cap B) \sum_{j=1}^{\infty} (1-p)^j \\ &= \frac{\mathbb{P}(X \in D' \cap B)}{p}.\end{aligned}$$

The first part is proved. Now, suppose that $X \sim \mathcal{U}(\mathcal{D})$ and $D \supseteq D'$. Then, for any, $B \in \mathcal{B}(\mathbb{R}^d)$,

$$D \supseteq D' \text{ and } B \cap D' \subseteq D$$

and next

$$p = \mathbb{P}(D) = \frac{\lambda(D')}{\lambda(D)} \text{ and } \mathbb{P}(B \cap D') = \frac{\lambda(B \cap D')}{\lambda(D)}.$$

and so:

$$\mathbb{P}(X_\tau \in B) = \frac{\lambda(B \cap D')}{\lambda(D')}.$$

Finally $X_\tau \in B) \sim \mathcal{U}(D')$. $\square$

3. The rejection method

Now, the next result will allow us to get an algorithm for generating a random vector X. Suppose that a random vector X with support $D \subseteq \mathbb{R}^d$ and with *pdf* f such that f is bounded by αg, where $\alpha > 0$ and g is a *pdf*. Let $U \sim \mathcal{U}(0, 1)$ independent of$X$. By part (A) of the theorem 1,

$$Z = (X, \alpha U g(X))$$

is uniformly distributed on

$$D_0 = \{x \in D, 0 \leq u \leq \alpha g(x)\}$$

and

$$\mathbb{P}(Z \in D_0) = 1.$$

We also have

$$D' = \{x \in D, 0 \leq u \leq f(x)\} \subseteq D_0.$$

Now, let Z_1, Z_2,... *iid* $\sim \mathcal{U}(D_0)$,

$$\mathbb{P}(Z \in D') = 1, \ for \ \ \tau = \inf\{j \geq 1, Z_j \in D')\}, \quad X_\tau \sim \mathcal{U}(D').$$

We have

$$\tau = \inf\{j \geq 1, (X_j, \alpha U_j g(X_j)) \in D'\}$$

i.e

$$\tau - 1 = \sup\{j \geq 1, (X_j, \alpha U_j g(X_j)) \in D'\}.$$

However,

$$\begin{aligned}(X_j, \alpha U_j g(X_j)) &\notin (x, u) \in D \times \mathbb{R}, 0 \leq u \leq f(x) \\ &= (\alpha U_j g(X_j) > f(X_j)) \\ &= \left(U_j(\frac{\alpha g(X_j)}{f(X_j)}) > 1\right).\end{aligned}$$

Setting

$$T_j = \frac{\alpha g(X_j)}{f(X_j)}$$

leads to

$$\tau = \inf\{j \geq 1, (X_j, U_j T_j \leq 1\}.$$

We get

$$Y_\tau = (X_\tau, \alpha U_\tau g(X_\tau)) \sim \mathcal{U}(D').$$

Let us justify that X_τ and U_τ are independent. For $B \in \mathcal{B}(\mathbb{R}^d)$, $A \in \mathcal{B}([0,1])$,

$$\begin{aligned}
\mathbb{P}(X_\tau \in B, U_\alpha \in A) &= \sum_{j=1}^{\infty} \mathbb{P}(X_\tau \in B, U_\tau \in A \big| (\tau = j))\ \mathbb{P}(\tau = j)\\
&= \sum_{j=1}^{\infty} \mathbb{P}(X_j \in B, U_j \in A)\mathbb{P}(\tau = j)\\
&= \sum_{j=1}^{\infty} \mathbb{P}(X_j \in B)\mathbb{P}(U_j \in A)\mathbb{P}(\tau = j)\\
&= \lambda(A)\sum_{j=1}^{\infty} \mathbb{P}(X_j \in B)\mathbb{P}(\tau = j)\\
&= \lambda(A)\mathbb{P}(X_\tau \in B)
\end{aligned}$$

But, for $B \in \mathbb{R}^d$, by using the last line,

$$\mathbb{P}(U_\tau \in A) = \mathbb{P}(U_\tau \in A, X_\tau \in \mathbb{R}^d) = \lambda(A).$$

Hence

$$\mathbb{P}(X_\tau \in B, U_\alpha \in A) = \mathbb{P}(X_\tau \in B)\mathbb{P}(U_\tau \in A).$$

Next, for $B \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{P}(X_\tau \in B) &= \mathbb{P}(X_\tau, \alpha U_\alpha g(X_\tau) \in B \times \mathbb{R})\\
&= \frac{\lambda_{d+1}(B \times \mathbb{R})}{\lambda_d(D')},
\end{aligned}$$

and

$$\begin{aligned}
\lambda_d(D') &= \int_{0 \leq u \leq f(x)} d\lambda_d(u) d\lambda(x)\\
&= \int_{x \in D} d\lambda_d(x) \int_0^{f(x)} d\lambda(u)\\
&= \int_D f(x) d\lambda_d(x) = 1.
\end{aligned}$$

We also have

$$\begin{aligned}\lambda_{d+1}(B \times \mathbb{R}) &= \int_{x\in B, 0\leq u\leq f(x)} d\lambda_d(x)d\lambda(u)\\ &= \int_B f(x)d\lambda_d(x).\end{aligned}$$

Finally,

$$\mathbb{P}(X_\tau \in B) = \int_B f(x)d\lambda_d(x)$$

i.e

$$d\mathbb{P}_{X_\tau}(B) = f(x)d\lambda_d(x).$$

From the results above, we have the following algorithm for for generating a random vector X of λ_d-*pdf* f of support D.

Algorithm (RM).

Repeat
  Step 1. Genenerate $X \sim g$, $U \sim \mathcal{U}(0,1)$, X and U independent,
  Step 2.set $T = \alpha\frac{g(X)}{f(X)}$,
Until $UT \leq 1$

Then return X. $\square$

4. Concluding remarks

To implement Algorithm 3 (page 67), we need $\alpha > 0$ and a *pdf* g such that αg dominated f, i.e., $0 \leq f \leq \alpha g$. Let us comment on them.

(a) The multiplicative constant α. By integrating both sides of $\alpha g \geq f$. we see that α is necessarily greater that one.

(b) Finding a *pdf* g is not enough to reach our goal of generating a random vector of *pdf* f. We also ensure that there exists an easy way to generate a random vector vector g.

So it is important to see that Algorithm 3 (page 67) is a transition method that works only if we are able to generate random vectors by other methods, at least from the bound g.

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