



# Scheffé's bounds for the approximation of the student law to the standard normal law

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**Abstract.** In this note, Scheffé's bounds for the approximation of the student law to the standard normal law are explicitly and entirely determined.

**Résumé.** Dans ce brève note, des bornes de Scheffé relatives à l'approximation de la loi de Student par la loi normale standard sont explicitement et entièrement déterminées.

**Key words:** weak convergence; theorem of Scheffé; student law; normal standard law; Sterling formula.

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## 1. Introduction

It is well-know that the student probability law  $t(n)$  converges to that of the standard normal one. The student probability law  $t(n)$  and the standard normal probability law  $\mathcal{N}(0, 1)$  have the respective probability density functions (*pdf*)

$$f_n(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2), \quad x \in \mathbb{R}$$

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The weak convergence of  $t(n)$  to  $\mathcal{N}(0,1)$  is justified by Scheffé's theorem (Billingsley (1968) or see Lo et al.(2016), Chapter 2, Section 5, page 69) which is,

$$(f_n \rightarrow f \text{ pointwise}) \Rightarrow (t(n) \rightsquigarrow \mathcal{N}(0,1)), \text{ as } n \rightarrow +\infty.$$

But the most powerful result in the results of Scheffé is the following

$$\sup_{B \in \mathcal{B}(\mathbb{R})} \left| \int_B f_n(x) dx - \int_B f(x) dx \right| = \frac{1}{2} \int_{\mathbb{R}} |f_n(x) - f(x)| dx \rightarrow 0,$$

where  $\mathcal{B}(\mathbb{R})$  is the class of Borel sets on  $\mathbb{R}$ . Here, we call bounds of

$$\Delta_n = \frac{1}{2} \int_{\mathbb{R}} |f_n(x) - f(x)|$$

as scheffé's bounds. When we apply this to sets  $B = ]-\infty, x]$ , we have the following result on the respective cumulative distribution functions *cdf*  $F_n$  and  $F$  of  $t(n)$  and  $\mathcal{N}(0,1)$  respectively :

$$\|F_n - F\|_{\infty} = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \leq \frac{1}{2} \int_{\mathbb{R}} |f_n(x) - f(x)| dx \equiv \frac{1}{2} \Delta_n. \quad (1)$$

So it is important to compute Scheffé bounds all weak laws of sequences of variables to a limiting random variables having all Radon-Nikodym derivatives to a measure.

In that context, we provide here Scheffé's bounds to that classical weak convergence.

## 2. The Scheffé's bounds

Given the notations introduced above, we have :

**Proposition 1.** For  $n \geq 4$ ,  $p$  being the integer part of  $(n/2)$ ,

for  $n = 2p$  even,  $p \geq 18$ ,

$$\|F_n - F\|_{\infty} \leq \frac{16.13}{(p-1)(1-16.13/(p-1))},$$

(b) for  $n = 2p + 1$  odd,  $p \geq 2$

$$\|F_n - F\|_{\infty} \leq \frac{1.359}{(p-1)(1-1.359/p)}.$$

### 3. Proof

Let

$$C_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)}. \quad (2)$$

We will give the rate of convergence of  $C_n$  to  $(\sqrt{2\pi})$  in Formula (11) below. Next, it is clear that for any fixed  $x \in \mathbb{R}$ ,  $\left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$  converges to  $\exp(-x^2/2)$ . So  $(f_n \rightarrow f \text{ pointwise})$  holds and thus  $(t(n) \rightsquigarrow \mathcal{N}(0, 1))$  by Scheffé's Theorem. Next we have the determine the rate of convergence given in

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \leq \frac{1}{2} \int_{\mathbb{R}} |f_n(x) - f(x)| dx \equiv \frac{1}{2} A_n dx. \quad (3)$$

We will proceed according to even and odd integers  $n$ . Let us begin by writing

$$|f_n(x) - f(x)| \leq \left| C_n - \frac{1}{\sqrt{2\pi}} \right| \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} + \frac{1}{\sqrt{2\pi}} \left| \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} - \exp(-x^2/2) \right| \quad (4)$$

For  $x$  fixed,  $\left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$  is increasing in  $n$  and converges to  $\exp(-x^2/2)$  and hence

$$\left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} \leq \exp(-x^2/2). \quad (5)$$

By integrating over  $x$ , we get

$$\forall n \geq 1, C_n \geq 1/\sqrt{2\pi}.$$

We may easily check the following facts

$$\text{For } n = 2p, \begin{cases} \Gamma\left(\frac{n}{2}\right) = (p-1)! \\ \Gamma\left(\frac{n+1}{2}\right) = \frac{(2p)!}{2^{(2p)}p!} \sqrt{\pi} \end{cases} \text{ and for } n = 2p+1, \begin{cases} \Gamma\left(\frac{n}{2}\right) = \frac{(2p)!}{2^{(2p)}p!} \sqrt{\pi} \\ \Gamma\left(\frac{n+1}{2}\right) = p! \end{cases}$$

Let us use the Stirling formula (See Valiron (1956), Feller (1968), or Lo (2016), appendix, for a full proof),

$$\forall k \geq 1, k! = \sqrt{2k\pi} k^k e^k e^\theta, |\theta| \leq \frac{1}{12k}.$$

Let  $n = 2p$ ,  $p \geq 1$ . Direct computations lead to

$$C_n = \frac{1}{2\pi} \frac{e^{\theta'}}{e \left(1 - \frac{1}{p}\right)^{p-1}} \left(1 + \frac{1}{p-1}\right)^{1/2} \quad (6)$$

with

$$\theta' \leq \frac{1}{12p} + \frac{1}{12(p-1)} + \frac{1}{24p}, \text{ (and hence) } 0 \leq \exp(\theta) - 1 \leq \frac{1}{4(p-1)}$$

We have for  $x \geq 0$ , (by dropping  $x^2/4$  below),

$$0 \leq (1+x)^{1/2} - 1 = ((1+x/1)^2 - x^2/4)^{1/2} - 1 \leq x/2$$

and applying this implies for  $p \geq 2$ .

$$D_n - 1 \equiv \left(1 + \frac{1}{p-1}\right)^{1/2} - 1 \leq \frac{2}{p-1} \quad (7)$$

From Ngom *et al.* (2016),

$$0 \leq B_n - 1 \equiv \left(e \left(1 - \frac{1}{p}\right)^{(p-1)} - 1\right) \leq 1.842 \left(\frac{1}{2p} + \frac{1}{p^2 \log p}\right).$$

Next for  $p \geq 2$  (and by using  $(1 - 1/p)^{-1} \geq 2e$ )

$$1 - \frac{1}{e \left(1 - \frac{1}{p}\right)^{p-1}} \leq \frac{2(B_n - 1)}{2} \leq 3.684e \left(\frac{1}{2p} + \frac{1}{p^2 \log p}\right). \quad (8)$$

By putting the previous facts together, we get

$$\left|C_n - \frac{1}{\sqrt{2\pi}}\right| \leq \frac{1}{\sqrt{2\pi}} (\exp(\theta') - 1) + |B_n - 1| + (D_n - 1) \quad (9)$$

where, by the mean value theorem,  $|\exp(\theta') - 1| \leq e\theta'$  since  $|\theta'| \leq 1$ ) and then, for  $p \geq 2$

$$\| \leq \frac{5.932e}{(p-1)\sqrt{2\pi}} = \frac{16.13}{(p-1)\sqrt{2\pi}} \quad (10)$$

hence we have

$$C_n \rightarrow \frac{1}{\sqrt{12\pi}} \text{ as } n \rightarrow +\infty. \quad (11)$$

On another side, by Formula (5),

$$\int_{\mathbb{R}} \left| \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} - \exp(-x^2/2) \right| dx \quad (12)$$

$$= \int_{\mathbb{R}} \left( \exp(-x^2/2) - \left(1 - \frac{x^2}{n}\right)^{-(n+1)/2} \right) dx \quad (13)$$

$$= \frac{1}{C_n} - \sqrt{2\pi} \quad (14)$$

$$= \left( \frac{1 - \sqrt{2\pi}C_n}{C_n} \right) \quad (15)$$

$$= \left( \frac{1 - \sqrt{2\pi}C_n - 1}{\sqrt{2\pi}} \right) \left( \frac{\sqrt{2\pi}}{C_n} \right). \quad (16)$$

But  $(1 - \sqrt{2\pi}C_n)/\sqrt{2\pi} = (2\pi)^{-1/2} - C_n$ . Hence

$$\int_{\mathbb{R}} \left| \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} - \exp(-x^2/2) \right| dx \leq \frac{\frac{1}{\sqrt{2\pi}} - C_n}{C_n}. \quad (17)$$

Finally, integrating Formula(4), we have

$$\int |f_n(x) - f(x)| dx \leq 2 \frac{\frac{1}{\sqrt{2\pi}} - C_n}{C_n} \quad (18)$$

and next, by using

$$C_n \geq \frac{1}{\sqrt{2\pi}} \left( 1 - \frac{16.13}{(p-1)} \right) \geq 0 \text{ for } p \geq 18,$$

We conclude that

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \leq \frac{16.13}{(p-1)(1 - 16.13/(p-1))}.$$

Computations are more simple for  $n = 2p + 1$ . Here we have

$$C_n = \frac{1}{\sqrt{2\pi}} \exp(\theta') \left( \frac{2p}{2p+1} \right).$$

Now

$$\left(\frac{2p}{2p+1}\right) = -\left(1 - \frac{1}{2p+1}\right)$$

and that the mean-value theorem, for  $p \geq 1$ ,

$$1 - H_n \equiv \left|1 - \left(1 - \frac{1}{2p+1}\right)\right| \leq \frac{\sqrt{2}}{2p+1}.$$

By using the same techniques we get

$$\left|C_n - \frac{1}{\sqrt{2\pi}}\right| \leq \frac{2e}{4p\sqrt{2\pi}} = \frac{1.359}{p}. \blacksquare$$

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