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Uniform Rates of Convergence of Some Representations of Extremes

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Abstract. Uniform convergence rates are provided for asymptotic representations of sample extremes. These bounds which are universal in the sense that they do not depend on the extreme value index are meant to be extended to arbitrary samples extremes in coming papers.

Résumé. Des vitesses de convergence uniforme, universelles en ce sens qu'elles ne dépendent pas de l'index extrémal, sont fournies pour des représentations asymptotiques des extrêmes d'échantillon. Ces bornes sont à étendre dans le cas général des extrêmes dans des articles à venir.

Key words: extreme value theory; weak convergence; rate of convergence, Schéffé's Theorem.
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1. Introduction

This paper presents a view on the rate of convergence of the univariate extremes of samples in a simple form. Rather than trying to handle the general problem for a distribution function in the extreme domain of attraction, we focus here on the simplest representations for which we give the most precise rates. The results are expected to serve as tools in general for the univariate case and later for the multivariate frame.

In a few words, the extreme value theory started with the univariate case, especially with independent data. Given a sequence of independent and identically distributed random variables $(X_n)_{n\geq 0}$ with common cumulative distribution function (*cdf*) and defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, the max-stability problem consists of finding possible limits in distribution of the sequences of partial maxima

$$M_n = max(X_1, ..., X_n),\ n \geq 1,$$

when appropriately centered and normalized. Precisely, we want to find non-random sequences $(a_n > 0)_{n\geq 1} \subset \mathbb{R}$ and $(b_n)_{n\geq 1} \subset \mathbb{R}$ such that $(M_n - b_n)/a_n$ converges in distribution to a random variable Z as $n \to +\infty$, denoted as

$$\frac{M_n - b_n}{a_n} \rightsquigarrow Z.\ \ (M)$$

This problem was solved around the middle of 20-th century with the contributions of many people, from whom we can cite Gnedenko (1943), Fisher and Tippet (1928), Fréchet (1927), etc. The following result is usually quoted as the Gnedenko (1943) result since he had the chance to close the characterization theorem. If Z is non-degenerate, that is : Z takes at least two different values, Formula (M) can hold only if Z is one of the three types in terms of its *cdf*:

the Fréchet type with parameter $\gamma \in \Gamma_1 = \{x \in \mathbb{R},\ x > 0\}$:

$$H_\gamma(x) \equiv \phi_\gamma(x) = \exp(-x^{-1/\gamma})1_{(x\geq 0)},$$

the Weibull type with parameter $\gamma \in \Gamma_2 = \{x \in \mathbb{R},\ x < 0\}$:

$$H_\gamma(x) \equiv \psi_\gamma(x) = \exp((-x)^{-1/\gamma})1_{(x<0)} + 1_{(x\geq 0)}$$

and the Gumbel type with parameter $\gamma \in \Gamma_0 = \{0\}$:

$$H_\gamma(x) \equiv \Lambda(x) = \exp(-e^{-x}), \quad x \in \mathbb{R}.$$

By type of distribution, we mean that any non-degenerate Z in Formula (M) have a *cdf* F_Z which is of the form $F_Z(x) = H_\gamma(Ax+B)$, where $0 < A \in \mathbb{R}$, $B \in \mathbb{R}$, and $\gamma \in \mathbb{R}$.

A modern account of the theory, including statistical estimation, can be found in Galambos (1958), de Haan (1970), de Haan and Ferreira (2006), Embrechts *et al.* (1997), Resnick (1987), Beirlant *et al.* (2014), etc. This theory is part of the general weak convergence which is thoroughly treated in Billingsley (1968) and van der Vaart and Wellner (1996). But all our needs in extreme value theory and in weak convergence are already gathered in Lo *et al.*(2016) and Lo (2016), which are currently cited in the paper.

We denote by $Z_1 = Fr(\gamma)$, $Z_2 = Wei(\gamma)$ and $Z_0 = \Lambda$ random variables admitting the above *cdf*'s respectively. We remark that the *cdf*'s H_γ are continuous, and the *cdf* of $(M_n - b_n)/a_n$ is $F(a_n x + b_n)^n, \ x \in \mathbb{R}$. By Theorem 3 in Chapter 2 and Point (5) in Chapter 4, Section 1 in Lo *et al.*(2016), (M) is equivalent to

$$R_n = \sup_{x \in \mathbb{R}} |F(a_n x + b_n)^n - H_\gamma(x)| \to 0 \ as \ n \to +\infty.$$

The general problem of the rate of convergence here is to find R_n or to have universal bounds for it.

As we already stressed, here we propose results to be used in the general case. To introduce this, suppose that we have a sequence of independent and uniformly distributed random variables $(U_n)_{n\geq 0}$ on $(0,1)$ and let us consider for each the $U_{1,n} = min(U_1, ..., U_n)$ for each $n \geq 1$ and define

$$Z_n(1,\gamma) = (nU_{1,n})^{-\gamma}, \ \gamma \in \Gamma_1; \quad Z_n(2,\gamma) = -(nU_{1,n})^{-\gamma}, \ \gamma \in \Gamma_2$$

and

$$Z_n(0,\gamma) = -\log(nU_{1,n}), \ \gamma \in \Gamma_0.$$

It is straightforward to show that for $\gamma \in \Gamma_i$, $Z_n(i,\gamma)$ weakly converges to $Fr(\gamma)$, to $Wei(\gamma)$ and to Λ accordingly to $i = 0$, $i = 1$ and $i = 2$. But the most important is that, based on Section 3.2 in Chapter 1 in Lo (2016) for example, if (M) holds with Z non-degenerate, we may always get the following asymptotic representation in distribution

$$\frac{M_n - b_n}{a_n} = Z_n(i,\gamma) + r_n, \ \text{ with } r_n \to 0 \ as \ n \to +\infty, \ (MG)$$

for some $i \in \{0,1,2\}$ with suitable sequences $(a_n > 0)_{n\geq 1} \subset \mathbb{R}$ and $(b_n)_{n\geq 1} \subset \mathbb{R}$.

Because of the general formula (MG), we think it is important to have a special study of the rate of convergence of the $Z_n(i,\gamma)$'s, which paves the way, further, for the handling of the general rate r_n.

This is what we are going to do exactly in Section 2.

2. Rate of convergence of $Z_n(i,\gamma)$

We are going to see that a universal bound will be guided by the following rate of convergence

$$\left(e\left(1-\frac{1}{n}\right)^{n-1}-1\right)$$

which is estimated as follows.

Lemma 1. *For all* $n \geq 2$

$$0 \leq \left(e\left(1-\frac{1}{n}\right)^{n-1}-1\right) \leq C_0\left(\frac{1}{2n}+\frac{1}{n^2 \log n}\right).$$

where $C_0 = exp(0.6106739) = 1.841673$.

Proof. It is given in the Appendix.

The main result of the paper is the following.

Theorem 1. *We have for all* $n \geq 2$, *for* $C_1 = 2 + C_0$,

$$\sup_{i\in\{0,1,2\}} \sup_{\gamma\in\Gamma_i} \sup_{x\in\mathbb{R}} |F_{Z_n(i,\gamma)}(x) - F_{Z_i}(x)| \leq \frac{C_1}{4n} + \frac{C_0}{2n^2 \log n}.$$

Proof. We first deal with the $Z_n(1,\gamma), \gamma > 0$. Set $\alpha = 1/\gamma$. We have

$$\forall(x \in \mathbb{R}),\ F_n(x) = F_{Z_n(1,\gamma)}(x) = \begin{cases} \left(1-\frac{1}{nx^\alpha}\right)^n & if \quad nx^\alpha > 1 \\ 0 & otherwise \end{cases}$$

and

$$\forall(x \in \mathbb{R}),\ f_n(x) = F'_{Z_n(1,\gamma)}(x) = \begin{cases} \alpha x^{-\alpha-1}\left(1-\frac{1}{nx^\alpha}\right)^{n-1} & if \quad nx^\alpha > 1 \\ 0 & otherwise. \end{cases}$$

Since, as $n \to +\infty$,

$$Z_n(1,\gamma) \rightsquigarrow Fr(1/\gamma),$$

we have, by Scheffé's Theorem (see for example Theorem 4, Chapter 3 in Lo *et al.*(2016)) :

$$\begin{aligned}\sup_{x\in\mathbb{R}} \left|F_{Z_n(1,\gamma)}(x) - \phi_\gamma(x)\right| &\leq \sup_{B \text{ measurable}} \int_B |f_n(x) - f(x)|\, dx\\ &= \frac{1}{2}\int |f_n(x) - f(x)|\, dx,\end{aligned}$$

with

$$\forall (x \in \mathbb{R}),\ f(x) = F^{'}_{Fr(1/\gamma)}(x) = \begin{cases} \alpha x^{-\alpha-1}e^{-x^{-\alpha}} & if \quad x > 0 \\ 0 & otherwise. \end{cases}$$

For short, we also write $F_{Fr(1/\gamma)} = F$. Now, we have to compute

$$2a_n(\alpha) = \int |f_n(x) - f(x)|\, dx. \tag{1}$$

We have

$$\begin{aligned}\int |f_n(x) - f(x)|\, dx &= \int_{[0,n^{-1/\alpha}]} |f_n(x) - f(x)|\, dx + \int_{]n^{-1/\alpha},+\infty[} |f_n(x) - f(x)|\, dx\\ &= F(n^{-1/\alpha}) + \int_{]n^{-1/\alpha},+\infty[} |f_n(x) - f(x)|\, dx. \end{aligned} \tag{2}$$

We also have

$$\int_{]n^{1/\alpha},+\infty[} |f_n(x) - f(x)|\, dx = \int_{]n^{-1/\alpha},+\infty[} \alpha x^{-\alpha-1}e^{-x^{-\alpha}} \left|1 - e^{x^{-\alpha}}\left(1 - \frac{1}{nx^\alpha}\right)^{n-1}\right| dx.$$

Now, let us study

$$g_{n,\alpha}(x) = e^{x^{-\alpha}}\left(1 - \frac{1}{nx^\alpha}\right)^{n-1}$$

on $]n^{-1/\alpha}, +\infty[$. Differentiating $g_{n,\alpha}(x)$ gives

$$g^{\prime}_{n,\alpha}(x) = \frac{\alpha}{n}e^{x^{-\alpha}}x^{-\alpha-1}\left(1 - \frac{1}{nx^\alpha}\right)^{n-2}(x^{-\alpha} - 1) \text{ on }]n^{-1/\alpha}, +\infty[.$$

For $n \geq 2$, $g'_{n,\alpha}(x)$ is positive on $]n^{-1/\alpha},\ 1]$ and negative on $[1,+\infty[$ so that $g_{n,\alpha}(x)$ is increasing $]n^{-1/\alpha}, 1]$ and decreasing on $[1,+\infty[$. Since for $n \geq 2$,

$$g_{n,\alpha}(1) = e(1-1/n)^{n-1} > 1$$

and, as $n \to +\infty$,

$$g_{n,\alpha}(1) \to 1^+ \text{ (i.e., by from above 1)}\ ,$$

we have a unique number $x_{n,\alpha} \in]n^{-1/\alpha}, 1]$ such that $g_{n,\alpha}(x_{n,\alpha}) = 1$ and

$$0 \leq g_{n,\alpha}(x) \leq 1 \text{ on }]n^{-1/\alpha}, x_{n,\alpha}]$$

and for all $x \geq n^{-1/\alpha}$,

$$1 \leq g_{n,\alpha}(x) \leq g_{n,\alpha}(1) = e(1-1/n)^{n-1}.$$

By exploiting the sign of $1 - g_{n,\alpha}(x)$ on $]n^{-1/\alpha}, 1]$ and $]1,\ +\infty]$ respectively, we have

$$\begin{aligned}
&\int_{]n^{-1/\alpha},+\infty[} \alpha x^{-\alpha-1} e^{-x^{\alpha}} \left| 1 - e^{x^{-\alpha}} \left(1 - \frac{1}{nx^{\alpha}}\right)^{n-1} \right| dx \\
\leq\ & \int_{n^{1/\alpha}}^{x_{n,\alpha}} \alpha x^{-\alpha-1} e^{-x^{-\alpha}} \left(1 - g_{n,\alpha}(x)\right) dx + \int_{]x_{n,\alpha},+\infty[} \alpha x^{-\alpha-1} e^{-x^{-\alpha}} \left(g_{n,\alpha}(x) - 1\right) dx \\
=\ & a_n(\alpha, 1) + a_n(\alpha, 2). \qquad (3)
\end{aligned}$$

Next, we have

$$\begin{aligned}
a_n(\alpha, 1) &= \int_{n^{-1/\alpha}}^{x_{n,\alpha}} f(x) dx - \int_{n^{-1/\alpha}}^{x_{n,\alpha}} \left(\left(1 - \frac{1}{nx^{\alpha}}\right)^{n}\right)' dx \\
&= F(x_{n,\alpha}) - F(n^{-1/\alpha}) - \left(1 - \frac{1}{nx_{n,\alpha}^{\alpha}}\right)^{n}. \qquad (4)
\end{aligned}$$

Combining all, this leads to

$$\begin{aligned}
a_n(\alpha, 2) &\leq \left(g_{n,\alpha}(1) - 1\right)\left(1 - F(x_{n,\alpha})\right) \\
&\leq \left(\left(e(1-1/n)^{n-1} - 1\right)\right) \equiv \alpha_n(3) \qquad (5)
\end{aligned}$$

In total, by putting Formulas 1-5, we get

$$\begin{aligned}
2\alpha_n(\alpha) &\leq \exp(-x_{n,\alpha}^{-\alpha}) - \left(1 - \frac{1}{n x_{n,\alpha}^{\alpha}}\right)^n + \alpha_n(3) \\
&\leq \exp(-x_{n,\alpha}^{-\alpha}) \left(1 - \left(\exp(x_{n,\alpha}^{-\alpha}) \left(1 - \frac{1}{n x_{n,\alpha}^{\alpha}}\right)^{n-1}\right) \left(1 - \frac{1}{n x_{n,\alpha}^{\alpha}}\right)\right) + \alpha_n(3) \\
&\leq \exp(-x_{n,\alpha}^{-\alpha}) \left(1 - g_{n,\alpha}(x_{n,\alpha}) \left(1 - \frac{1}{n x_{n,\alpha}^{\alpha}}\right)\right) + \alpha_n(3) \ (L2) \\
&\leq \exp(-x_{n,\alpha}^{-\alpha}) \frac{1}{n x_{n,\alpha}^{\alpha}} + \alpha_n(3), \\
&\leq \frac{1}{n} \left(x_{n,\alpha}^{-\alpha} \exp(-x_{n,\alpha}^{-\alpha})\right) + \alpha_n(3),
\end{aligned}$$

where we identified $g_{n,\alpha}(x_{n,\alpha})$, which is equal to one, in Line 2 in the above group of formulas. Now, the term between the big parentheses is bounded by the supremum of the function $\ell(x) = x^{-\alpha} e^{-x^{-\alpha}}$ on $(0,1)$ whose derivative, $\alpha x^{-\alpha-1} e^{-x^{-\alpha}}(x^{-\alpha} - 1)$, is non-negative on $(0,1)$. Hence our bound is $\ell(1) = 1/e$. We conclude that for $n \geq 2$

$$2\alpha_n(\alpha) \leq \frac{1}{n} + \left(e(1 - 1/n)^{n-1} - 1\right).$$

By using Lemma 1, we finally get

$$\alpha_n(\alpha) \leq \frac{1}{2n} + C_0 \left(\frac{1}{4n} + \frac{2}{n^2 \log n}\right).$$

From there, we use Lemma 1 to conclude the proof for $Z_n(1,\gamma)$. To finish for the other cases, it is enough to see that we have the following two formulas:

$$\sup_{x \in \mathbb{R}} |F_{Z_n(2,\gamma)}(x) - F_{Z_2}(x)| = \sup_{x \in \mathbb{R}} |F_{Z_n(1,-1/\gamma)}(-x^{-1}) - F_{Z_1}(-x^{-1})|$$

and

$$\sup_{x \in \mathbb{R}} |F_{Z_n(0,\gamma)}(x) - F_{Z_0}(x)| = \sup_{x \in \mathbb{R}} |F_{Z_n(1,1)}(e^x) - F_{Z_1}(e^x)|,$$

which puts an end to the proof.

Remark. The bounds provided here are to be used in the general theory of a coming paper.

3. Appendix

Proof. Fix $n \geq 2$. We have

$$(n-1)\log(1-1/n) = -1 + \sum_{k\geq 1}\left(\frac{1}{k} - \frac{1}{k+1}\right)\frac{1}{n^k},$$

which in turn implies

$$1 \leq g_{n,\ \alpha}(1) = e\left(1-\frac{1}{n}\right)^{n-1} = \exp\left(\sum_{k\geq 1}\frac{1}{k(k+1)n^k}\right).\ (F1)$$

By the comparison methods between series with monotonic terms, say $a_n = a(n)$ with the appropriate improper Riemann integral of $a(x)$, we have

$$\begin{aligned}\sum_{k\geq 1}\frac{1}{k(k+1)n^k} &\leq \frac{1}{2n} + \int_2^{+\infty}\frac{dx}{n^x}\\ &\leq \frac{1}{2n} + \frac{1}{n^2\log n}.\ (F2)\end{aligned}$$

By combining (F1) and (F2), we have, for $0 \leq \theta \leq 1$,

$$\begin{aligned}0 \leq \left(e\left(1-\frac{1}{n}\right)^{n-1} - 1\right) &\leq \exp\left(\frac{1}{2n} + \frac{1}{n^2\log n}\right) - 1\\ &\leq \left(\frac{1}{2n} + \frac{1}{n^2\log n}\right)\exp\left(\theta\left(\frac{1}{2n} + \frac{1}{n^2\log n}\right)\right)\\ &\leq exp(0.6106739)\left(\frac{1}{2n} + \frac{1}{n^2\log n}\right).\end{aligned}$$

References

Beirlant, J., Goegebeur, Y. Teugels, J. (2004). *Statistics of Extremes Theory and Applications.* Wiley. (MR2108013)

Billingsley, P. (1968). *Convergence of Probability measures.* John Wiley, New-York.

de Haan, L. (1970). *On regular variation and its application to the weak convergence of sample extremes.* Mathematical Centre Tracts, **32**, Amsterdam. (MR0286156)

de Haan, L. and Ferreira A. (2006). *Extreme value theory : An introduction.* Springer. (MR2234156)

Embrechts, P., Kűppelberg C. and Mikosh T. (1997). Modelling extremal events for insurance and Finance. Springer Verlag.

Fisher, R. and Tippet, L. (1928) Limiting Forms of the Frequency Distribution of the Largest or Smallest Member of a Sample. *Proceedings of the Cambridge Philosophical Society*, 24, 180-190.

Fréchet, M. (1927). Sur la loi de probabilité de l'écart maximum. *Annales de la Société Polonaise de Mathématique*, 1927, 6, 93-116.

Galambos, J. (1985). The Asymptotic theory of Extreme Order Statistics. *Wiley*, Nex-York. (MR0489334)

Gnedenko, B. (1943) Sur la distribution limite du terme maximum d'une série aléatoire. *Annals of Mathematics*, 44, 423-453

Lo, G.S., Ngom, M. and Kpanzou T.A. (2016). Weak Convergence (IA). Sequences of random vectors. SPAS Books Series. Saint-Louis, Senegal - Calgary, Canada. Doi : 10.16929/sbs/2016.0001. Arxiv : 1610.05415. ISBN : 978-2-9559183-1-9

Lo, G.S. (2016). Univariate Theory of Extreme Value Theory and Statistical Estimation in the Extreme Value Domain. SPAS Books Series. Saint-Louis, Senegal - Calgary, Canada. Doi : 10.16929/sbs/2016.0009. Arxiv : 1610.05415. ISBN : ISBN 978-2-9559183-9-5

Resnick, S.I. (1987). *Extreme Values, Regular Variation and Point Processes*. Springer-Verlag, New-York. (MR0900810)

van der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and Empirical Processes With Applications to Statistics*. Springer.