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A simple proof of the theorem of Sklar

Gane Samb Lo

[1] LERSTAD, Gaston Berger University, Saint-Louis, Sénégal.
[2] Affiliated to LSTA, Université Pierre et Marie Curie, Paris, France
[3] Associated with African University of Sciences and Technology, AUST, Abuja, Nigeria
1178 Evanston Drive NW, T3P 0J9, Calgary, Alberta, Canada

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Abstract. In this note we provide a quick proof of the Sklar's Theorem on the existence of copulas by using the generalized inverse functions as in the one dimensional case, but a little more sophisticated.

Résumé. Dans cette note, nous donnons une preuve rapide the Théorème de Sklar sur l'existence de copule en utilisant les fonctions inverses généralisées en dimension une, mais dans une forme plus poussée.

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Corresponding author gane-samb.lo@ugb.edu.sn, gslo@aust.edu.ng, ganesamblo@ganesamblo.net

1. Introduction

Let us begin to define generalized functions. Let $[a, b]$ and $[c, d]$ be non-empty intervals of $\mathbb{R}$ and let $G : [a, b] \mapsto [c, d]$ be a non-decreasing mapping such that

$$c = \inf_{x \in [a,b]} G(x), \quad (L11)$$
$$d = \sup_{x \in [a,b]} G(x). \quad (L12)$$

Since G is a non-decreasing mapping, this ensures that

$$a = \inf\{x \in \mathbb{R},\ G(x) > c\}, \quad (L13)$$
$$b = \sup\{x \in \mathbb{R}, G(x) < d\}. \quad (L14)$$

If $x = a$ or $x = b$ is infinite, the value of G at that point is meant as a limit. If $[a, b]$ is bounded above or below in $\mathbb{R}$, G is extensible on $\mathbb{R}$ by taking $G(x) = G(a+)$ for $x \leq a$ and $G(x) = G(b - 0)$ for $x \geq b$. As a general rule, we may consider G simply as defined on $\mathbb{R}$. In that case, $a = lep(G)$ and $b = uep(G)$ are called *lower end-point* and *upper end-point* of G.

The generalized inverse function of G is given by

$$\forall u \in [lep(G), uep(G)],\ G^{-1}(u) = \inf\{x \in \mathbb{R},\ G(x) \geq u\}.$$

The properties of G^{-1} have been thoroughly studied, in particular in Billingsley (1968), Resnick (1987). The results we need in this paper are gathered and proved in Lo *et al.*(2016a) or in Lo *et al.*(2016b) (Chapter 4, Section 1) and reminded as below.

Lemma 1. *Let G be a non-decreasing right-continuous function with the notation above. Then G^{-1} is left-continuous and we have*

$$\forall u \in [c, d],\ G(G^{-1}(u)) \geq u\ (A) \text{ and } \forall x \in [a, b],\ G^{-1}(G(x)) \leq x\ (B)$$

and

$$\forall x \in [lep(G), uep(G)],\ G^{-1}(G(x) + 0) = x. \quad (1)$$

Proof. The proof of Formulas (A) and (B) are well-known and can be found in the cited books above. Let us prove Formula (1) for any $x \in [a, b]$.

On one side, we start by the remark that $G^{-1}(G(x) + 0)$ is the limit of $G^{-1}(G(x) + h)$ as $h \searrow 0$. But for any $h > 0$, $G^{-1}(G(x) + h)$ is the infimum of the set of $y \in [a, b]$ such that $G(y) \geq G(x) + h$. Any these y satisfies $y \geq x$. Hence $G^{-1}(G(x) + 0) \geq x$.

On the other side $G(x+h) \searrow G(x)$ by right-continuity of G, and by the existence of the right-hand limit of the non-decreasing function $G^{-1}(\circ)$, $G^{-1}(G(x+h)) \searrow G^{-1}(G(x)+0)$. Since $G^{-1}(G(x+h)) \leq x+h$ by Formula (B), we get that $G^{-1}(G(x)+0) \leq x$ as $h \searrow 0$. The proof is complete. □

The remainder of the note will focus on distribution functions (*df*'s) and copulas on $\mathbb{R}^d$. For an introduction to *df*'s, we refer to Lo(2017) (Chapter 11) and for copulas to Nelsen (2006).

Let us also remind the definition of a distribution function on $\mathbb{R}^d$, $d \geq 1$. A mapping $\mathbb{R}^d \mapsto \mathbb{R}$ is a *df* if and only if :

(**DF1**) F is right-continuous

and

(**DF2**) F assigns to non-negative volumes to any cuboid $]a,b]$ (with $a=(a_1,...,a_d) \leq b=(b_1,...,b_d)$ meaning $a_i \leq b_i$, for all $1 \leq i \leq d$), that is :

$$\Delta F(a,b) = \sum_{\varepsilon \in \{0,1\}^d} (-1)^{s(\varepsilon)} F(b+\varepsilon * (a-b)) \geq 0, \tag{2}$$

where

$$(x,y) * (X,Y) = (x_1X_1, x_2X_2, ..., y_kY_k),$$

$\varepsilon = (\varepsilon_1, \cdots, \varepsilon_d)$ runs over $\{0,1\}^d$ and $s(\varepsilon) = \varepsilon_1 + \cdots + \varepsilon_d$.

It becomes a cumulative distribution function *cdf* or simply a probability distribution function if F satisfies the third limit conditions (which by the way ensure that F is non-negative) :

(**DF3a**)

$$\lim_{\exists i, 1 \leq i \leq k, t_i \rightarrow -\infty} F(t_1, ..., t_k) = 0,$$

and

(**DF3b**)

$$\lim_{\forall i, 1 \leq i \leq k, t_i \rightarrow +\infty} F(t_1, ..., t_k) = 1.$$

Now we come to copulas. By definition, a copula on $\mathbb{R}^d$ is a *cdf* C whose marginal *cdf*'s defined by, for $1 \leq i \leq d$,

$$\mathbb{R} \ni s \mapsto C_i(s) = C\left(+\infty, ..., +\infty, \underbrace{s}_{i-th\ argument}, +\infty, ..., +\infty\right),$$

are all equal to the $(0,1)$-uniform *cdf* which in turn is defined by

$$x \mapsto x1_{[0,1[} + 1_{[1,+\infty[},$$

and we may also write, for all $s \in [0,1]$,

$$C_i(s) = C\left(1, ..., 1, \underbrace{s}_{i-th\ argument}, 1, ..., 1\right) = s. \tag{3}$$

Based on the notation above, the theorem of Sklar (1959) is :

Theorem 1. *For any cdf F on $\mathbb{R}^d$, $d \geq 1$, there exists a copula C on $\mathbb{R}^d$ such that*

$$\forall x \in \mathbb{R}^d,\ F(x) = C(F_1(x), ..., F_d(x)). \tag{4}$$

This theorem is now among the most important tools in Statistics since it allows to study the dependence between the components of a random vector through the copula, meaning that the intrinsic dependence does not depend on the margins.

In the frame we have set, the proof of the Sklar's theorem is easily proved in a few lines, using the properties of generalized inverse.

2. A proof of the Sklar's Theorem

Define for $s = (s_1, s_2, \cdots, s_d) \in [0,1]^d$,

$$C(s) = F(F_1^{-1}(s_1+0), F_2^{-1}(s_2+0), \cdots, F_d^{-1}(s_d+0)). \tag{5}$$

It is immediate that C assigns non-negative volumes to cuboids of $[0,1]^d$, since according to Condition (DF2), Formula (2) for C derives from the same for F where the arguments are the form $F_i^{-1}(\circ+0)$, $1 \leq i \leq d$.

Also C is right-continuous since F is right-continuous as well as each $F_i^{-1}(\circ+0)$, $1 \leq i \leq d$. By passing, this explains why we took the right-limits because the $F_i^{-1}(\circ)$'s are left-continuous.

Finally, by combining Formulas (1) and (5), we get the conclusion of Sklar in Formula (4). The proof is finished. □

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