# A parallel treatment of semi-continuous functions with left and right-continuous functions and some application in pedagogy

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**Abstract.** Left and right-continuous functions play an important role in Real analysis, especially in Measure Theory and Integration on the real line and in Stochastic processes indexed by a continuous real time. Semi-continuous functions are also of major interest in the same way. This paper aims at presenting a useful handling of semi-continuous function in parallel with the way left or right continuous functions are treated. For example, a lower or upper continuous function shares the property that it is measurable if it fails to be upper or lower continuous at most at a number of countable points with a left or right-continuous when it fails to be left or right-continuous at most at a number of countable points. As a final result, the comparison between the Riemann and the Lebesgue integrals on compact sets is done in a very comfortable and comprehensive way. The frame used here is open to more further sophistication.

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**Résumé**. Les fonctions continues à droite et/ou à gauche jouent un rôle important en Analyse réelle, surtout en Théorie de la Mesure et de l'Intégration, et en processus stochastiques indexés par un temps continu réel. Les fonctions semi-continues aussi jouent un rôle tout aussi important. Ce papier est l'occasion d'offrir une présentation des fonctions semi-continues parallèlement aux propriétés des fonctions continues à droite et/ou à gauche. Par exemple, une fonction semi-continue inférieurement ou supérieurement partage la propriété qu'elle est encore mesurable si elle est semi-continue inférieurement ou supérieurement en dehors d'un ensemble au plus dénombrable de points. Finalement, l'approche utilisée rend plus compréhensible et plus confortable la comparaison entre les intégrales de Riemman et de Lebesgue sur un interval compact de la droite réelle. L'approche est ouverte à de futurs développements.

## 1. Introduction

Left and right continuous functions play an important role in real analysis, in particular in measure theory on the real line. In this paper, we proceed to a study of semi-continuous functions following the way we do with let or continuous functions. Though the material used here is classical, the present treatment lead to a very comfortable handling of the mentioned materials and their use in a number of problems of modern analysis.

The ideas summarized here come from our teaching of measure theory and integration and are closely related to Borel functions and the Lebesgue and Riemann integration. One will feel the Baire's ideas behind all parts of the text. We hope that this note might be considered as an input in the classical measure theory teaching. This note might be written in a compact and short form. But we rather give details to allow students to read it.

To define semi-continuous numerical functions, recall the superior and inferior limit of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ :

$$f^*(x) = \limsup_{y \to x} f(y) = \lim_{\varepsilon \downarrow 0} \sup\{f(y), y \in ]x - \varepsilon, x + \varepsilon[\}$$

and

$$f_*(x) = \liminf_{y \to x} f(y) = \lim_{\varepsilon \downarrow 0} \inf\{f(y), y \in ]x - \varepsilon, x + \varepsilon[\}$$

As well we may define left superior and inferior of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ :

$$f^{*,\ell}(x) = \limsup_{y \to x} f(y) = \lim_{\varepsilon \downarrow 0} \sup\{f(y), y \in ]x - \varepsilon, x[\}$$

and

$$f_{*,\ell}(x) = \liminf_{y \to x} f(y) = \lim_{\varepsilon \downarrow 0} \inf\{f(y), y \in ]x - \varepsilon, x[\}$$

and the right superior and inferior of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ :

$$f^{*,r}(x) = \limsup_{y \to x} f(y) = \lim_{\varepsilon \downarrow 0} \sup\{f(y), y \in ]x, x + \varepsilon[\}$$

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and

$$f_{*,r}(x) = \liminf_{y \to x} f(y) = \lim_{\varepsilon \downarrow 0} \inf\{f(y), y \in ]x, x+\varepsilon[\}.$$

These definitions are possible because the extrema we used in them are either non-decreasing or non-increasing as $\varepsilon \downarrow 0$ and the limits as $\varepsilon \downarrow 0$ are infima or suprema in $\overline{\mathbb{R}}$. For example, for the definition of $f^*(x)$ and $f_*(x)$, $\sup\{f(y), y \in ]x-\varepsilon, x+\varepsilon[\}$ is non-increasing as $\varepsilon \downarrow 0$ and $\inf\{f(y), y \in ]x-\varepsilon, x+\varepsilon\}$ is non-decreasing $\varepsilon \downarrow 0$.

The following properties also also known, whenever the expressions make sense :

(1) $f$ is respectively continuous at $x \in \mathbb{R}$, or $f$ is right-continuous, or $f$ is right continuous at $x$ if and only if, we respectively have $f^*(x) = f_*(x)$, or $f^{*,\ell}(x) = f_{*,\ell}(x)$, or $f^{*,r}(x) = f_{*,r}(x)$.

(2) For any $x \in$, $(-f(x))^* = -f_*(x)$, $(-f(x))^{*,\ell} = -f_{*,\ell}(x)$, $(-f(x))^{*,r} = -f_{*,r}(x)$.

In the sequel the functions $f^*, f_*, f^{*,\ell}, f_{*,\ell}, f^{*,r}$ and $f_{*,r}$ are defined on a set $I \subset \mathbb{R}$, whenever they exist, by the following graphs :

$$x \hookrightarrow f^*(x), \ x \hookrightarrow f_*(x), \ x \hookrightarrow f^{*,\ell}(x), x \hookrightarrow f_{*,\ell}(x), x \hookrightarrow f^{*,r}(x).$$

We have the following definitions.

**Definition 1.** We define

**(1)** A function $f : \mathbb{R} \mapsto \mathbb{R}$ is upper semi-continuous (usc) on an open interval $I$ of $\mathbb{R}$ if and only if, for any $x \in I$, $f(x) = f^*(x)$. It is lower semi-continuous on $I$ (lsc) if and ony if , for any $x \in I$, $f(x) = f_*(x)$.

**(2)** A function $f : \mathbb{R} \mapsto \mathbb{R}$ is upper left-semi-continuous (ulsc) on an open interval $I$ of $\mathbb{R}$ if and only if, for any $x \in I$, $f(x) = f^{*,\ell}(x)$. It is lower left-semi-continuous (llsc) on $I$ if and only if, for any $x \in I$, $f(x) = f_{*,\ell}(x)$.

(1) A function $f : \mathbb{R} \mapsto \mathbb{R}$ is upper right-semi-continuous (ursc) on an open interval $I$ of $\mathbb{R}$ if and only if , for any $x \in I$, $f(x) = f^{*,r}(x)$. It is lower right-semi-continuous (lrdc) on $I$ if and only if, for any $x \in I$, $f(x) = f_{*,r}(x)$.

**Immediate remark**. A right-continuous function needs to be lower right-semi-continuous and upper right-semi-continuous. But a lower or upper semi-continuous function is not necessarily right or left-continuous. So using *ulsc*, *ursc*, *llsc* or *lrdc* functions may extend the working classes in Analysis beyond continuous or left or right-continuous functions.

The definitions below are automatically extensible to numerical functions defined on metric spaces. But let just see that we still have the classical definitions of semi-continuous functions on $\mathbb{R}$. We have

**Proposition 1.** *A function $f : \mathbb{R} \mapsto \mathbb{R}$ is upper semi-continuous (usc) at $x \in I$, where $I$ is an open interval of $\mathbb{R}$ if and only if*

$$\forall \varepsilon > 0, \ \exists \eta > 0, \ (|y - x| < \eta \ and \ y \in I) \Longrightarrow f(y) \leq f(x) + \varepsilon \tag{1}$$

*A function $f : \mathbb{R} \mapsto \mathbb{R}$ is lower semi-continuous (usc) at $x \in I$, where $I$ is an open interval of $\mathbb{R}$ if and only if*

$$\forall \varepsilon > 0, \ \exists \eta > 0, \ (|y - x| < \eta \ and \ y \in I) \Longrightarrow f(x) - \varepsilon \leq f(y).$$

**Proof**. It is enough to prove only the first assertion, since the second may be derived from the first and vice-versa. So, we have to prove that (1) is equivalent to $f^*(x) = f(x)$. Suppose that (1) holds. Then for any $\varepsilon > 0$, there exists $\eta > 0$ such that $y \in ]x - \eta, x + \eta[$ implies that $f(y) \leq f(x) + \varepsilon$. This means that for any $0 < \delta \leq \eta$,

$$f(x) \leq f_\delta^*(x) = \sup\{]x - \delta, x + \delta[\} \leq f(x) + \varepsilon.$$

We get, as $\delta \downarrow 0$, for any $\varepsilon > 0$,

$$f(x) \leq f^*(x) \leq f(x) + \varepsilon.$$

We get $f^*(x) = f(x)$ by letting $\varepsilon \downarrow 0$. Let us prove the reverse implication, *ad absordium*. Suppose that (1) is false. Then there exists $\varepsilon > 0$ such that for all for all $\eta > 0$, there exists $y(\eta) \in y \in ]x - \eta, x + \eta[$ such that $f(y(\eta)) > f(x) + \varepsilon$, which means that

$$f_\eta^*(x) > f(x) + \varepsilon.$$

By letting $\eta \downarrow 0$, we get

$$f^*(x) \geq f(x) + \varepsilon,$$

so that the equality $f^*(x) = f(x)$ does not holds.

We are going to see that these functions have similar properties with right and left-continuous functions.

## 2. Semi-continuity and measurability

In this section the most useful result is of the following type

**Theorem 1.** *For any function $f$, $f^*$ is upper semi-continuous whenever it is finite and $f_*$ is lower semi-continuous it is finite.*

From this result, we get the following proposition.

**Proposition 2.** *Upper and lower semi-continuous functions are measurable.*

**Proof of Proposition 2**. Let $f$ be upper semi-continuous. Set for each $n \geq 1$,

$$D_n = \{k2^{-n}, k \in \mathbb{Z}, n \geq 1\}.$$

and

$$f_n(x) = \left( \sum_{k=-\infty}^{k=+\infty} \sup\{f(z), z \in ]k2^{-n}, (k+1)2^{-n}[\} 1_{]k2^{-n},(k+1)2^{-n}[}(x) \right) 1_{D_n^c} + \sum_{s \in D_n} f(s) 1_{\{s\}}(x).$$

For each $n \geq 1, f_n$ is measurable. Let us show that as $n \to +\infty$, for each $x \in \mathbb{R}$, $f_n(x) \to f(x)$. We have two cases.

If

$$x \in D = \bigcup_{n \geq 1} D_n,$$

we have, as $n \to +\infty$,

$$f_n(x) = f(x) \to f(x).$$

If

$$x \notin D = \bigcup_{n \geq 1} D_n,$$

then for $n \geq 1$, there exists $k = k(n,x)$ such that $x \in ]k2^{-n}, (k+1)2^{-n}[$. Then there exists $\varepsilon(n) > 0$ such that for any $0 < \varepsilon \leq \varepsilon(n)$, we have

$$]x - \varepsilon, x + \varepsilon[ \subset ]k(n,x)2^{-n}, (k(n,x)+1)2^{-n}[, \textbf{ (A)}$$

and next,

$$\sup\{f(z), z \in ]x - \varepsilon, x + \varepsilon[\} \leq \sup\{f(z), z \in [k2^{-n}, (k+1)2^{-n}[\}. \textbf{ (B)}$$

As well, for any $\eta > 0$, for $2^{-n} < \eta$, for $\varepsilon(n) > 0$ such that (A) holds, for any $n \geq 1$, for any

$$x \in ]k(n,x)2^{-n}, (k(n,x)+1)2^{-n}[ \subset ]x - \eta, x + \eta[,$$

we have

$$\sup\{f(z), z \in ]k(n,x)2^{-n}, (k(n,x)+1)2^{-n}[\} \subset \leq \{f(z), z \in ]x - \eta, x + \eta[\}. \textbf{ (C)}$$

By combining (A) and (B), we have for $\eta > 0$, $2^{-n} < \eta$, for $\varepsilon(n) > 0$ such that (A) holds, for any $0 < \varepsilon \leq \varepsilon(n)$,

$$f^{*,\varepsilon} \leq f_n \leq f^{*,\eta}.$$

Now let $\varepsilon \downarrow 0$ and only after, let $n \to +\infty$ to have

$$f^* \leq \liminf_{n\to+\infty} f_n(x) \leq \limsup_{n\to+\infty} f_n(x) \leq f^{*,\eta}.$$

Finally, let $\varepsilon \downarrow 0$ to get

$$f_n(x) \to f^* = f(x).$$

Then, we have $f_n(x) = f(x) \to f(x)$ as $n \to +\infty$. We conclude that $f$ is measurable.

This proves the measurability of an upper semi-continuous. To prove this for a lower semi-continuous, use the relation that for an lower semi-continuous $f$, $-f$ is an upper semi-continuous and the conclusion is immediate.□

Before we go further, let us mention that the same proof is valid for the following corollary.

**Corollary 1.** *Let $f$ be real-valued function defined on $I = [a, b]$ or $I = \mathbb{R}$ and for each $n$, let $I_{i,n} = (x_{i,n}, x_{i+1,n})$, $i \in J(n)$, be consecutive intervals, with non zero length, which partitions $I$ such that*

$$\sup_{i\in J(n)} |x_{i+1,n} - x_{i+1,n}| \to 0 \ as \ n \to +\infty.$$

*Denote $D = \cup_n \cup_{i\in J(n)} \{x_{i,n}\}$. We have for any $x \notin D$,*

$$H_n(x) = \sum_i \sup\{f(z), z \in ]x_{i,n}, x_{i+1,n}[\} 1_{]x_{i,n}, x_{i+1,n}[}(x) \to f^*(x)$$

*and*

$$h_n(x) = \sum_i \inf\{f(z), z \in ]x_{i,n}, x_{i+1,n}[\} 1_{]x_{i,n}, x_{i+1,n}[}(x) \to f_*(x).$$

**Proof of Theorem 1**. Set $g = f^*$. Let us show that $g^* = g$. Let $x$ be fixed. By definition for $\varepsilon > 0$,

$$g^*_\varepsilon(x) = \sup\{g(z), z \in ]x - \varepsilon, x + \varepsilon[\} = \sup\{f^*(z), z \in ]x - \varepsilon, x + \varepsilon[\}$$

Suppose that $g^*(x)$ is finite. Let us use the characterization of the supremum : for all $\eta > 0$, there exists $z_0 \in ]x - \varepsilon, x + \varepsilon[$ such that

$$g^*_\varepsilon(x) - \eta < f^*(z_0) < g^*_\varepsilon(x). \quad (C1)$$

Since $z_0 \in ]x - \varepsilon, x + \varepsilon[$, there exists $\varepsilon_0 > 0$ such that $z_0 \in ]z_0 - \varepsilon_0, z_0 + \varepsilon_0[ \subset ]x - \varepsilon, x + \varepsilon[$. Remind that $f_h^*(z_0) \downarrow f^*(z_0)$ as $h \downarrow 0$. Then for $0 < h \leq \varepsilon_0$,

$$f_h^*(z_0) = \sup\{f(z), z \in ]z_0 - h, z_0 + h[\} \leq \sup\{f(z), z \in ]x - \varepsilon, x + \varepsilon[\} = f_\varepsilon^*(x). \quad (C2)$$

By combining (C1) and (C2) we get

$$g_\varepsilon^*(x) - \eta < f^*(x).$$

Let $\eta \downarrow 0$ and next $\varepsilon \downarrow 0$ to get

$$g^*(x) \leq f^*(x). \qquad \text{(INEQ)}$$

Suppose that this inequality is strict, that is

$$g^*(x) < f^*(x).$$

So, we can find $\eta > 0$ such that

$$g^*(x) < f^*(x) - \eta.$$

Since $g_\varepsilon^*(x) \downarrow g^*(x)$, there exists $\varepsilon_0$ such that

$$g^*(x) \leq g_{\varepsilon_0}^*(x) < f^*(x) - \eta,$$

that is

$$g^*(x) \leq \sup\{g(z), z \in ]x - \varepsilon_0, x + \varepsilon_0[\} < f^*(x) - \eta,$$

that is also

$$g^*(x) \leq \sup\{f^*(z), z \in ]x - \varepsilon_0, x + \varepsilon_0[\} < f^*(x) - \eta.$$

But

$$x \in ]x - \varepsilon_0, \; x + \varepsilon_0[$$

and then

$$f^*(x) \leq \sup\{f^*(z), z \in ]x - \varepsilon_0, \; x + \varepsilon_0[\} < f^*(x) - \eta.$$

We arrive at the conclusion

$$f^*(x) < f^*(x) - \eta.$$

This is absurd. Hence the inequality (INEQ) is an equality, that is

$$g^*(x) = f^*(x).$$

Now suppose that $g^*(x)$ is infinite. This means that for any $A > 0$, we can find in any interval $]x - \varepsilon, x - \varepsilon[$, a point $z(\varepsilon)$ such that

$$g(z(\varepsilon)) > A.$$

Now fix $A$ and $\varepsilon > 0$. Consider $z(\varepsilon)$ such that the last inequality holds. Since $z(\varepsilon) \in \ ]x - \varepsilon, \ x - \varepsilon[$, there exists $r_0 > 0$ such that

$$z(\varepsilon) \in \ ]z(\varepsilon) - r_0, z(\varepsilon) + r_0[ \subset ]x - \varepsilon, x + \varepsilon[.$$

So, since

$$f_r^*(z(\varepsilon)) \downarrow g(z(\varepsilon)) = f_r^*(z(\varepsilon)) > A,$$

then for $r$ small enough

$$f_r^*(z(\varepsilon)) > A/2,$$

that is

$$\sup\{f(z), z \in \ ]z(\varepsilon) - r, z(\varepsilon) + r[\} > A/2$$

and again there will be a $u \in z \in \ ]z(\varepsilon) - r, z(\varepsilon) + r[$ such that

$$f(u) > A/4.$$

If $r$ is taken small enough such that $r \leq r_0$, we have found a point $u \in \ ]x - \varepsilon, \ x - \varepsilon[$ such that

$$f(u) > A/4.$$

We have proved that for any $A > 0$, for any $\varepsilon > 0$, we can find a point in $u \in \ ]x - \varepsilon, \ x - \varepsilon[$ such that

$$f(u) > A/4$$

and hence

$$f_\varepsilon^*(x) = \sup\{f^*(z), z \in \ ]x - \varepsilon, ]x + \varepsilon[\} \geq A/4.$$

Let $\varepsilon \downarrow 0$ and next $A \uparrow +\infty$ to get that

$$g(x) = f^*(x) = +\infty.$$

Thus

$$g^*(x) = g(x) = +\infty.$$

We have finished and we proved that

$$g^* = g.$$

**Conclusion** : if $g^*$ is finite, then it is upper semi-continuous. We similarly use that if $g_*$ is finite, then it is lower semi-continuous , by exploiting the formula $(-f)^* = -f_*$.

We get the following useful result

**Corollary 2.** *Any function $f$ which is lower semi-continuous except at a countable number of points and any function $g$ which is upper semi-continuous except at a countable number points are measurable.*

**Proof**. We give the proof only for a lower semi-continuous function $f$. The other case is derived from the application of the results of the first case to the opposite of $f$.

Let us denote the countable $D$ set on which $f$ is not lower semi-continuous. We have

$$f = f_* \text{ on } D^c$$

and then

$$f = f_* 1_{D^c} + \sum_{s \in D} f(s) 1_{\{s\}}. \tag{2}$$

By Theorem 1 and 2, $f_*$ is lower semi-continuous. Hence by (2), $f$ is measurable.

*2.1. Application to a rephrasing of the classical comparison between Riemann and Lebesgue integrals*

**Proposition 3.** *Let $[a,b]$ be a bounded and nonempty compact interval of $\mathbb{R}$ and $\lambda$ be the Lebesgue measure on $[a,b]$. Let $f : [a,b] \mapsto \mathbb{R}$ be a bounded function by $M$. Then $f$ is Riemann integrable if and only if $f$ is almost $\lambda$-a.e. continuous, and its Riemann integral is equal to its Lebesgue integral.*

**Proof of Proposition 3**. In that proof, for all rules using measure theory and integration, we refer to Lo (2017b). Consider the finite non-empty and bounded interval $]a,b]$ of $\mathbb{R}$ and a bounded, say by $M$, real-valued function $f$ defined on $]a,b]$. For each $n \geq 1$, consider a subdivision of $]a,b]$, which partitions $]a,b]$ into the $\ell(n)$ sub-intervals, in the form,

$$]a,b] = \sum_{i=0}^{\ell(n)-1} ]x_{i,n}, x_{i+1,n}].$$

with modulus

$$m(\pi_n) = \max_{0 \leq i \leq \ell(n)} (x_{i+1,n} - x_{i+1,n}) \to 0 \ as \ n \to +\infty.$$

Define for each $n \geq 0$, for each $i$, $0 \leq i \leq \ell(n) - 1$

$$m_{i,n} = \inf\{f(z), x_{i,n} \leq z < x_{i+1,n}\} \text{ and } M_{i,n} = \sup\{f(z), x_{i,n} \leq z < x_{i+1,n}\}$$

$$h_n = \sum_{i=0}^{\ell(n)-1} m_{i,n} 1_{]x_{i,n}, x_{i+1,n}]} \text{ and } H_n = \sum_{i=0}^{\ell(n)-1} M_{i,n} 1_{]x_{i,n}, x_{i+1,n}]}.$$

and

$$D = \bigcup_k \{x_{0,n}, x_{1,n}, ..., x_{\ell(n),n}\}.$$

Remark that $f^*$ and $f_*$ are bounded by $M$ and $D$ is countable. For a fixed $x$ in $x \in [a,b] \setminus D$, for any $n \geq 1$, there exists $i$, $0 \leq i(n) \leq \ell(n)-1$ such that $x \in ]x_{i(n),n}, x_{i(n)+1,n}[$. Then, we can find $\varepsilon > 0$ such that

$$]x - \varepsilon, x + \varepsilon[ \subseteq ]x_{i(n),n}, x_{i(n)+1,n}[.$$

It is clear that $x \in [a,b] \setminus D$

$$h_n(x) \leq f_*^\varepsilon(x).$$

and

$$H_n(x) \geq f_*^\varepsilon(x).$$

By letting $\varepsilon \downarrow 0$, we get

$$h_n(x) \leq f_*(x) \leq f^* \leq H_n(x). \tag{3}$$

So (3) holds for any $x \in [a,b] \setminus D$. Now, let $\eta > 0$. For any fixed $n \geq 1$, we may use the characterization of the suprema and the infima on $\mathbb{R}$, and show that the Lebesgue integral of $h_n$ and $H_n$ can be approximated to Riemann sums in the form

$$\int H_n d\lambda = \sum_{i=0}^{\ell(n)-1} M_{i,n}(x_{i+1,n} - x_{i,n}) < \sum_{i=0}^{\ell(n)-1} f(d_{i,n})(x_{i+1,n} - x_{i,n}) + \eta$$

and

$$\int h_n d\lambda = \sum_{i=0}^{\ell(n)-1} m_{i,n}(x_{i+1,n} - x_{i,n}) > \sum_{i=0}^{\ell(n)-1} f(c_{i,n})(x_{i+1,n} - x_{i,n}) - \eta.$$

By putting together the previous facts, we get

$$\sum_{i=0}^{\ell(n)-1} f(c_{i,n})\,(x_{i+1,n} - x_{i,n}) - \eta$$

$$\leq \int_{[a,b]\setminus D} f_*(x)d\lambda$$

$$\leq \int_{[a,b]\setminus D} f^*(x)d\lambda$$

$$\leq \sum_{i=0}^{\ell(n)-1} f(d_{i,n})(x_{i+1,n} - x_{i,n}) + \eta,$$

where we use that $D$ is countable and then, is a $\lambda$-null set. By letting $\eta \to 0$, we get

$$\int_{[a,b]\setminus D} f_*(x)\,d\lambda(x) = \int_{[a,b]\setminus D} f^*(x)\,d\lambda(x).$$

Thus, we have that

$$f_* = f^* = f$$

outside a countable subset $[a,b] \setminus D$, then outside a $\lambda$-null subset of $[a,b]$ and

$$\int_a^b f(x)\,d\lambda(x) = \int_{[a,b]} f(x)d\lambda(x). \tag{4}$$

Now suppose that $f$ is *a.e.* continuous, that is $f_* = f^* = f \ \lambda - a.e.$ Consider a sequence of Riemann sums for which the sequence of modulus tends to zero with $n$ :

$$S_n = \sum_{i=0}^{\ell(n)-1} f(c_{i,n})(F(x_{i+1,n}) - F(x_{i,n})).$$

Let us denote the $\lambda$-null set $H = (f^\star \neq f_\star)$We have

$$\int h_n d\lambda \leq S_n \leq \int H_n d\lambda$$

Denoting $H = [a,b] \setminus D$, we have

$$\int h_n 1_H d\lambda \leq S_n \leq \int H_n h_n 1_H d\lambda$$

By Corollary 1 above, $h_n 1_H \to f_* 1_H$ and $H_n 1_H \to f^* 1_H$ as $n \to +\infty$. Since $|f| \leq M$ which is $\lambda$-integrable on $[a,,b]$, we may apply the Dominated Convergence Theorem to get

$$\int f_* 1_H d\lambda \leq \liminf_{n \to +\infty} S_n \leq \limsup_{n \to +\infty} S_n \leq \int f_* 1_H d\lambda.$$

From there, we have

$$\liminf_{n \to +\infty} S_n = \limsup_{n \to +\infty} S_n = \int f_* 1_H d\lambda = \int f_* 1_H d\lambda = \int f d\lambda.$$

So, all Riemann sums converge to

$$I = \int f d\lambda \in \mathbb{R},$$

when the modulus goes to zero. Then $f$ is Riemann integrable and Formula 4 holds again.

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