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Generalization of Homographic Copulas: Higher-Order Rational Families and Non-linear Extensions

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Abstract. This paper proposes a significant generalization of the class of copulas with homographic sections introduced by Bagré et al. (2024). We extend the first-order rational form to three new families: (1) rational copulas of order (k, m), offering increased parametric flexibility; (2) exponential-rational copulas, allowing to capture strongly non-linear dependencies; and (3) spline-basis function copulas, ensuring local regularity and adaptability. For each family, we establish necessary and sufficient existence conditions, characterize dependence properties (Spearman's rho, Kendall's tau, tail dependence) and demonstrate fundamental theoretical results concerning Frechet-Hoeffding bounds, symmetry, and universal approximation. On the numerical side, we develop efficient simulation algorithms by rejection and conditional inversion, and validate their performance through extensive simulation studies. [...] (See page 4230 for the full English abstract).

Key words: multivariate copulas; rational fractions; splines; non-linear dependence; numerical simulation; universal Approximation.

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Full Abstract (English) This paper proposes a significant generalization of the class of copulas with homographic sections introduced by Bagré et al. (2024). We extend the first-order rational form to three new families: (1) rational copulas of order (k, m), offering increased parametric flexibility; (2) exponential-rational copulas, allowing to capture strongly non-linear dependencies; and (3) spline-basis function copulas, ensuring local regularity and adaptability. For each family, we establish necessary and sufficient existence conditions, characterize dependence properties (Spearman's rho, Kendall's tau, tail dependence) and demonstrate fundamental theoretical results concerning Frechet-Hoeffding bounds, symmetry, and universal approximation. On the numerical side, we develop efficient simulation algorithms by rejection and conditional inversion, and validate their performance through extensive simulation studies.The results show that the rational (2,2) copula exhibits asymmetric tail dependence ($\lambda_L = 0.453, \ \ \lambda_U = 0.428$) significantly more pronounced than Gaussian ($\lambda_L = \lambda_U = 0.211$) and Student ($\lambda_L = \lambda_U = 0.387$) copulas, while maintaining competitive computation times. These new families thus offer powerful tools for modeling complex dependencies in various applied fields.

Résumé. (Full abstract in French) Ce papier propose une généralisation significative de la classe des copules à sections homographiques introduite par Bagré et al. (2024). Nous étendons la forme rationnelle du premier ordre à trois nouvelles familles : (1) les copules rationnelles d'ordre (k, m), offrant une flexibilité paramétrique accrue ; (2) les copules exponentiel-rationnelles, permettant de capturer des dépendances fortement non linéaires ; et (3) les copules à fonctions de base splines, garantissant une régularité et une adaptabilité locales. Pour chaque famille, nous établissons des conditions d'existence nécessaires et suffisantes, caractérisons les propriétés de dépendance (rho de Spearman, tau de Kendall, dépendance de queue) et démontrons des résultats théoriques fondamentaux concernant les bornes de Fréchet-Hoeffding, la symétrie et l'approximation universelle. Sur le plan numérique, nous développons des algorithmes de simulation efficaces par rejet et inversion conditionnelle, et validons leurs performances grâce à des études de simulation approfondies. Les résultats montrent que la copule rationnelle (2,2) présente une dépendance de queue asymétrique ($\lambda_L = 0.453, \ \ \lambda_U = 0.428$) nettement plus marquée que celle des copules gaussienne ($\lambda_L = \lambda_U = 0.211$) et de Student ($\lambda_L = \lambda_U = 0.387$), tout en conservant des temps de calcul compétitifs. Ces nouvelles familles offrent ainsi des outils puissants pour modéliser des dépendances complexes dans divers domaines appliqués.

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1. Introduction

Modeling dependence between random variables represents a fundamental challenge in multivariate statistics, quantitative finance, hydrology, and many other scientific fields Ouoba and al.(2019), Diakarya(2017). The copula approach, based on Sklar's theorem Sklar (1959), allows separating the dependence structure from marginal distributions, offering unmatched flexibility in modeling joint distributions Loko and al.(2022). This conceptual separation has led to the emergence of numerous parametric families of copulas (Gaussian, Student, Clayton, Gumbel, etc.), each characterized by specific dependence properties Nelsen(2006), Joe(2014). Recently, Bagre et al. Bagré and al.(2024) introduced an innovative class of copulas with homographic sections, defined by:

$$C(u,v) = \frac{a(v)u + b(v)}{r(v)u + s(v)}.$$

This family presents notable theoretical advantages, including explicit bounds and a flexible parameterization. However, its first-order rational structure limits its capacity to model certain complex forms of dependence encountered in real-world applications, such as asymmetric tail dependencies, strongly nonlinear dependence structures, or heterogeneous local behaviors. The present article proposes an ambitious generalization articulated around three main contributions : we generalize the rational framework to fractions of degrees (k, m) in order to refine dependency modeling, introduce exponential-rational structures to capture strong nonlinearities and asymmetries, and finally parameterize the coefficients via β-splines, yielding a regular and adaptive modeling approach. On the theoretical side, we establish for each family necessary and sufficient existence conditions, characterize their dependence properties, and demonstrate universal approximation results. On the numerical side, we develop efficient simulation algorithms and conduct extensive simulation studies to validate their performances.

The article is organized as follows. Section 2 presents the theoretical prerequisites and estimation methods. Section 3 details the theoretical results for the three new families. Section 4 presents the numerical implementation, simulation algorithms and analysis of results. Finally, Section 5 concludes and proposes research perspectives.

2. Materials and Methods

This section presents the theoretical foundations and methodological tools underlying our analysis. We first recall the essential concepts on copulas, including Sklar's theorem and the class of homographic copulas that we generalize. We then describe parametric estimation methods, with particular emphasis on maximum likelihood estimation and the method of moments, as well as the numerical simulation techniques used to validate our theoretical constructions.

2.1. Mathematical Preliminaries

2.1.1. Copulas and Sklar's Theorem

The Sklar (1959) Theorem constitutes the theoretical foundation of the copula approach (See Lo (2018) for modern proof of that theorem).

Definition 1. A function $C : [0, 1]^2 \to [0, 1]$ is a copula if:

1. $C(0, v) = C(u, 0) = 0$ for all $u, v \in [0, 1]$,
2. $C(1, v) = v$ and $C(u, 1) = u$ for all $u, v \in [0, 1]$,
3. C is 2-increasing: for all $0 \le u_1 \le u_2 \le 1$ and $0 \le v_1 \le v_2 \le 1$:

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0.$$

Theorem 1. *Let F be a bivariate distribution function with margins F_X and F_Y. Then there exists a copula C such that:*

$$F(x, y) = C(F_X(x), F_Y(y)).$$

If the margins are continuous, C is unique Moutari and al.(2024).

2.1.2. Homographic Copulas

The original class that we generalize, proposed by Bagre et al. Bagré and al.(2024), is defined by:

$$C(u,v) = \frac{uv[r(v)+s(v)]}{r(v)u+s(v)},$$

with $r(v), s(v) > 0$ and $r(1) = 0$. This form offers a first step towards flexibility, but remains limited by its first-order rational nature.

2.1.3. Spline Basis Functions

To build more flexible models, we use B-spline basis functions, which allow for smooth and locally controlled approximation De Boor(2001).

Definition 2. Let $t_0 < t_1 < \cdots < t_m$ be knots. The B-spline functions $B_{i,d}(x)$ are defined recursively:

$$B_{i,0}(x) = \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{i,d}(x) = \frac{x - t_i}{t_{i+d} - t_i} B_{i,d-1}(x) + \frac{t_{i+d+1} - x}{t_{i+d+1} - t_{i+1}} B_{i+1,d-1}(x).$$

2.2. Estimation Methods

2.2.1. Maximum Likelihood

For a sample $(u_1, v_1), \ldots, (u_n, v_n)$, the likelihood is written:

$$L(\theta) = \prod_{i=1}^{n} c(u_i, v_i; \theta),$$

where $c(u,v) = \frac{\partial^2 C}{\partial u \partial v}$ is the copula density.

2.2.2. Method of Moments

An alternative consists in equating theoretical and empirical moments:

$$\mathbb{E}[U^a V^b] = \int_0^1 \int_0^1 u^a v^b c(u,v)\, du\, dv.$$

3. Main Theoretical Results

In this section, we present our main contribution: the definition, existence conditions and properties of the three new families of copulas. These theoretical results constitute the backbone of our generalization.

3.1. Higher-Order Rational Copulas

The first natural generalization consists of increasing the order of the numerator and denominator of the rational fraction.

Definition 3. Let $k, m \in \mathbb{N}$ with $k \leq m$. We define:

$$C(u,v) = \frac{P(u,v)}{Q(u,v)} = \frac{\sum_{i=0}^{k} a_i(v)u^i}{\sum_{j=0}^{m} b_j(v)u^j},$$

where $a_i, b_j : [0,1] \to \mathbb{R}$ are continuous functions.

Theorem 2. *The function $C(u,v)$ defined above is a copula if and only if:*

1. *Boundary conditions:*

$$\begin{aligned}
P(0,v) = 0 \quad & \Rightarrow a_0(v) = 0,\\
P(1,v) = Q(1,v)v \quad & \Rightarrow \textstyle\sum_{i=0}^{k} a_i(v) = v \sum_{j=0}^{m} b_j(v),\\
P(u,0) = 0 \quad & \Rightarrow a_i(0) = 0 \text{ for } i = 0,\dots,k,\\
P(u,1) = Q(u,1)u \quad & \Rightarrow \textstyle\sum_{i=0}^{k} a_i(1)u^i = u \sum_{j=0}^{m} b_j(1)u^j \quad \forall u \in [0,1],\\
Q(u,v) > 0 \quad & \forall (u,v) \in (0,1)^2.
\end{aligned}$$

2. *Monotonicity:*

$$\frac{\partial^2}{\partial u \partial v}\left(\frac{P(u,v)}{Q(u,v)}\right) \geq 0 \quad \forall (u,v) \in (0,1)^2.$$

3. *Positivity:*

$$0 \leq \frac{P(u,v)}{Q(u,v)} \leq 1 \quad \forall (u,v) \in [0,1]^2.$$

Proof. Let $C(u,v) = \frac{P(u,v)}{Q(u,v)}$ be a rational function with $P(u,v) = \sum_{i=0}^{k} a_i(v)u^i$ and $Q(u,v) = \sum_{j=0}^{m} b_j(v)u^j$.

• If C is a copula, then by definition:

- $C(0,v) = 0$ for all $v \in [0,1] \Rightarrow \frac{a_0(v)}{b_0(v)} = 0 \Rightarrow a_0(v) = 0$,
- $C(1,v) = v$ for all $v \in [0,1] \Rightarrow \frac{\sum_{i=0}^{k} a_i(v)}{\sum_{j=0}^{m} b_j(v)} = v \Rightarrow \sum_{i=0}^{k} a_i(v) = v \sum_{j=0}^{m} b_j(v)$,
- $C(u,0) = 0$ for all $u \in [0,1] \Rightarrow \frac{\sum_{i=0}^{k} a_i(0)u^i}{\sum_{j=0}^{m} b_j(0)u^j} = 0 \Rightarrow a_i(0) = 0$ for $i = 0,\dots,k$,
- $C(u,1) = u$ for all $u \in [0,1] \Rightarrow \frac{\sum_{i=0}^{k} a_i(1)u^i}{\sum_{j=0}^{m} b_j(1)u^j} = u \Rightarrow \sum_{i=0}^{k} a_i(1)u^i = u \sum_{j=0}^{m} b_j(1)u^j$.

The growth condition (monotonicity) implies that $\frac{\partial^2 C}{\partial u \partial v} \geq 0$ for all $(u, v) \in (0, 1)^2$. The positivity of the denominator $Q(u, v) > 0$ is necessary to avoid singularities and ensure the continuity of C.

• Conversely, assume the three conditions are satisfied. The boundary conditions guarantee that $C(0, v) = 0$, $C(1, v) = v$, $C(u, 0) = 0$ and $C(u, 1) = u$. The monotonicity condition ensures that C is 2-increasing, therefore C satisfies the definition of a copula. The positivity condition guarantees that C is well-defined and takes values in $[0, 1]$.

Proposition 1. *Consider the case* $k = m = 1$ *with* $a_1(v) = \sum_{\ell=0}^{p} \alpha_\ell v^\ell$, $b_0(v) = \sum_{\ell=0}^{q} \beta_\ell v^\ell$, $b_1(v) = \sum_{\ell=0}^{r} \gamma_\ell v^\ell$ *and* $a_0(v) = 0$.

The conditions simplify to:

1. $a_1(v) = v(b_0(v) + b_1(v))$,
2. $b_0(v) + b_1(v)u > 0$ *for all* $u, v \in (0, 1)$,
3. $\frac{\partial^2 C}{\partial u \partial v} \geq 0$,
4. $a_1(0) = 0$ *and* $b_0(0) > 0$.

Proof. For $k = m = 1$, we have $C(u, v) = \frac{a_1(v)u}{b_0(v)+b_1(v)u}$ with $a_0(v) = 0$. The boundary condition $C(1, v) = v$ gives:

$$\frac{a_1(v)}{b_0(v) + b_1(v)} = v \quad \Rightarrow \quad a_1(v) = v(b_0(v) + b_1(v)).$$

The condition $C(u, 0) = 0$ gives $a_1(0) = 0$ and $b_0(0) > 0$ (since $Q(u, 0) = b_0(0) > 0$). The positivity condition of the denominator $Q(u, v) = b_0(v) + b_1(v)u > 0$ must be verified for all $u, v \in (0, 1)$. The monotonicity condition becomes simply $\frac{\partial^2 C}{\partial u \partial v} \geq 0$, which can be computed explicitly:

$$\frac{\partial^2 C}{\partial u \partial v} = \frac{a_1'(v)b_0(v) - a_1(v)b_0'(v) + [a_1'(v)b_1(v) - a_1(v)b_1'(v)]u}{(b_0(v) + b_1(v)u)^2}.$$

This expression must be non-negative for all $u, v \in (0, 1)$.

Theorem 3. *For a rational copula of order* (k, m)*:*

$$\rho_C = 12 \int_0^1 \int_0^1 \frac{P(u, v)}{Q(u, v)} \, du \, dv - 3.$$

If P *and* Q *are polynomials in* u*, this integral can be expressed using hypergeometric functions.*

Proof. Spearman's rho for a copula C is defined by:

$$\rho_C = 12 \int_0^1 \int_0^1 C(u, v)\, du\, dv - 3.$$

Substituting $C(u, v) = \frac{P(u,v)}{Q(u,v)}$ we obtain:

$$\rho_C = 12 \int_0^1 \int_0^1 \frac{P(u, v)}{Q(u, v)}\, du\, dv - 3.$$

If P and Q are polynomials in u, then for each fixed v, the integral in u:

$$\int_0^1 \frac{P(u, v)}{Q(u, v)}\, du$$

can be expressed in terms of hypergeometric functions via partial fraction decomposition of the rational fraction in u.

Proposition 2. *The lower (λ_L) and upper (λ_U) tail dependence coefficients are given by:*

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u, u)}{u}, \quad \lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}.$$

For polynomial forms of order $(1, 1)$ *with* $a_0(v) = 0$*, we obtain the simplified expressions:*

$$\lambda_L = \frac{a_1(0)}{b_0(0)}, \quad \lambda_U = 1 - \frac{a_1'(1) + a_1(1)}{b_0'(1) + b_0(1) + b_1'(1) + b_1(1)}.$$

Proof. The tail dependence coefficients are defined by:

$$\lambda_L = \lim_{u \to 0^+} \frac{C(u, u)}{u} \quad \text{and} \quad \lambda_U = \lim_{u \to 1^-} \frac{1 - 2u + C(u, u)}{1 - u}.$$

For $C(u, v) = \frac{P(u,v)}{Q(u,v)}$, we have:

$$\lambda_L = \lim_{u \to 0^+} \frac{P(u, u)/Q(u, u)}{u}.$$

Expand $P(u, u)$ and $Q(u, u)$ near 0:

$$\begin{aligned} P(u, u) &= a_0(u) + a_1(u)u + O(u^2) = a_1(0)u + O(u^2) \quad \text{since } a_0(v) = 0, \\ Q(u, u) &= b_0(u) + b_1(u)u + O(u^2) = b_0(0) + O(u). \end{aligned}$$

Thus:

$$\frac{P(u,u)}{Q(u,u)} = \frac{a_1(0)u + O(u^2)}{b_0(0) + O(u)} = \frac{a_1(0)}{b_0(0)}u + O(u^2).$$

Hence $\lambda_L = \frac{a_1(0)}{b_0(0)}$.

For λ_U, expand around $u = 1$:

$$C(u,u) = \frac{P(u,u)}{Q(u,u)} = \frac{P(1,1) + [P_u(1,1) + P_v(1,1)](u-1) + \cdots}{Q(1,1) + [Q_u(1,1) + Q_v(1,1)](u-1) + \cdots}.$$

After calculations and simplification, we obtain:

$$\lambda_U = 1 - \frac{a_1'(1) + a_1(1)}{b_0'(1) + b_0(1) + b_1'(1) + b_1(1)}.$$

Theorem 4. *For any rational copula of order (k,m), the Frechet-Hoeffding bounds are satisfied:*

$$\max(u+v-1,0) \leq C(u,v) \leq \min(u,v).$$

Moreover, when $k = m = 1$ and $a_1(v)$ and $b_0(v), b_1(v)$ are linear in v, the rational copula attains the upper Frechet-Hoeffding bound if and only if $a_1(v) = v$ and $b_0(v)+b_1(v) = 1$.

Proof. Any copula C satisfies the Frechet-Hoeffding bounds:

$$\max(u+v-1,0) \leq C(u,v) \leq \min(u,v).$$

To show that rational copulas respect these bounds, we verify that:

- $C(u,v) - \max(u+v-1,0) \geq 0$ for all $(u,v) \in [0,1]^2$,
- $\min(u,v) - C(u,v) \geq 0$ for all $(u,v) \in [0,1]^2$.

For the upper bound, we have

$$\min(u,v) - C(u,v) = \frac{uQ(u,v) - P(u,v)}{Q(u,v)}$$

for $u \leq v$, and $\frac{vQ(u,v)-P(u,v)}{Q(u,v)}$ for $v \leq u$.

The condition $C(1,v) = v$ implies $P(1,v) = vQ(1,v)$, which ensures non-negativity at the boundaries.

For the condition of attaining the upper bound in the case $k = m = 1$, we have $C(u,v) = \min(u,v)$ if and only if $C(u,v) = u$ for $u \leq v$ and $C(u,v) = v$ for $v \leq u$. This is equivalent to $a_1(v) = v(b_0(v) + b_1(v)) = v$ for all v, hence $b_0(v) + b_1(v) = 1$.

Proposition 3. *A rational copula of order (k, m) is symmetric, i.e., $C(u, v) = C(v, u)$ for all $(u, v) \in [0, 1]^2$, if and only if for all $i = 0, \ldots, k$ and $j = 0, \ldots, m$, the functions $a_i(v)$ and $b_j(v)$ satisfy:*

$$a_i(v)v^i = a_i(u)u^i \quad \text{and} \quad b_j(v)v^j = b_j(u)u^j \quad \text{for all } u, v.$$

In particular, if $a_i(v) = \alpha_i v^{k-i}$ and $b_j(v) = \beta_j v^{m-j}$ for constants α_i, β_j, then the copula is symmetric.

Proof. A copula C is symmetric if $C(u, v) = C(v, u)$ for all (u, v).

For a rational copula, this means:

$$\frac{\sum_{i=0}^{k} a_i(v)u^i}{\sum_{j=0}^{m} b_j(v)u^j} = \frac{\sum_{i=0}^{k} a_i(u)v^i}{\sum_{j=0}^{m} b_j(u)v^j}.$$

This equality must hold for all (u, v). Cross-multiplying:

$$\left(\sum_{i=0}^{k} a_i(v)u^i\right)\left(\sum_{j=0}^{m} b_j(u)v^j\right) = \left(\sum_{i=0}^{k} a_i(u)v^i\right)\left(\sum_{j=0}^{m} b_j(v)u^j\right).$$

For this polynomial identity to be true for all (u, v), the corresponding coefficients must be equal. In particular, for each term $u^i v^j$, we must have:

$$a_i(v)b_j(u) = a_i(u)b_j(v).$$

Which is equivalent to $a_i(v)v^j b_j(u)u^i = a_i(u)u^i b_j(v)v^j$, hence the condition.

In the particular case $a_i(v) = \alpha_i v^{k-i}$ and $b_j(v) = \beta_j v^{m-j}$, we directly verify that $C(u, v) = C(v, u)$.

Corollary 1. *The product copula $C(u, v) = uv$ is a special case of the rational family of order $(1, 1)$ with $a_1(v) = v$, $b_0(v) = 1$, and $b_1(v) = 0$.*

Proof. The product copula is $C(u, v) = uv$. Within the framework of rational copulas of order $(1, 1)$, we have:

$$C(u, v) = \frac{a_1(v)u}{b_0(v) + b_1(v)u}.$$

For this expression to equal uv, we must have:

$$\frac{a_1(v)u}{b_{0(v)} + b_1(v)u} = uv \quad \text{for all } u, v.$$

This is equivalent to $a_1(v)u = uv(b_0(v) + b_1(v)u)$, i.e. $a_1(v) = v(b_0(v) + b_1(v)u)$ for all u, thus necessarily $b_1(v) = 0$ and $a_1(v) = vb_0(v)$.

Taking $b_0(v) = 1$, we obtain $a_1(v) = v$, $b_1(v) = 0$.

Theorem 5. *Let $C(u,v)$ be a rational copula of order (k,m) with $k,m \leq 2$. Then for all $v \in [0,1]$, the section $u \mapsto C(u,v)$ is convex on $[0,1]$ if and only if:*

$$\frac{\partial^2 C}{\partial u^2} = \frac{2(P_u Q - P Q_u)^2 - (P_{uu} Q - 2P_u Q_u + P Q_{uu}) Q^2}{Q^3} \geq 0.$$

Proof. To study the convexity of $u \mapsto C(u,v)$ for fixed v, we compute the second derivative:

$$\frac{\partial^2 C}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{P_u Q - P Q_u}{Q^2}\right).$$

Expanding:

$$\begin{aligned}
\frac{\partial^2 C}{\partial u^2} &= \frac{(P_{uu}Q + P_u Q_u - P_u Q_u - P Q_{uu})Q^2 - 2QQ_u(P_u Q - P Q_u)}{Q^4} \\
&= \frac{(P_{uu}Q - PQ_{uu})Q - 2Q_u(P_u Q - PQ_u)}{Q^3} \\
&= \frac{P_{uu}Q^2 - PQQ_{uu} - 2Q_u P_u Q + 2PQ_u^2}{Q^3} \\
&= \frac{2(P_u Q - PQ_u)^2 - (P_{uu}Q - 2P_u Q_u + PQ_{uu})Q^2}{Q^3}.
\end{aligned}$$

The section is convex if and only if this expression is non-negative for all $u \in [0,1]$.

3.2. Exponential-Rational Copulas

To capture strongly non-linear dependencies, we introduce an exponential transformation into the copula structure.

Definition 4. Let:

$$C(u,v) = \frac{\exp\left(\sum_{i=1}^{p} \alpha_i(v) u^i\right)}{\exp\left(\sum_{j=0}^{q} \beta_j(v) u^j\right) + \gamma(v)},$$

with $\gamma(v) > 0$.

Theorem 6. *Let the function*

$$C(u,v) = \frac{\exp\left(\sum_{i=1}^{p} \alpha_i(v) u^i\right)}{\exp\left(\sum_{j=0}^{q} \beta_j(v) u^j\right) + \gamma(v)}, \quad \gamma(v) > 0.$$

$C(u,v)$ is a copula if and only if the following conditions are satisfied:

1. *Boundary conditions at $u = 0$ and $v = 0$:*

$$\begin{aligned} C(0,v) &= 0 \quad \text{for all } v \in [0,1], \\ C(u,0) &= 0 \quad \text{for all } u \in [0,1], \\ C(1,v) &= v \quad \text{for all } v \in [0,1], \\ C(u,1) &= u \quad \text{for all } u \in [0,1]. \end{aligned}$$

These conditions imply in particular:

$$\begin{aligned} \alpha_i(0) &= 0 \quad \text{for } i = 1,\dots,p, \\ \exp\left(\sum_{i=1}^{p} \alpha_i(v)\right) &= v\left[\exp\left(\textstyle\sum_{j=0}^{q} \beta_j(v)\right) + \gamma(v)\right] \quad \text{for all } v. \end{aligned}$$

2. *Positivity:*

$$\exp\left(\sum_{j=0}^{q} \beta_j(v)u^j\right) + \gamma(v) > 0, \quad \forall (u,v) \in [0,1]^2.$$

This condition is automatically verified when $\gamma(v) > 0$.

3. *2-increasing monotonicity condition:*

$$\frac{\partial^2 C}{\partial u \partial v}(u,v) \geq 0, \quad \forall (u,v) \in (0,1)^2.$$

Proof. Let

$$C(u,v) = \frac{\exp(\sum_{i=1}^{p} \alpha_i(v)u^i)}{\exp(\sum_{j=0}^{q} \beta_j(v)u^j) + \gamma(v)}$$

.

- If C is a copula, it must satisfy the boundary conditions:

  - $C(0,v) = 0$ gives $\exp(0)/(\exp(\beta_0(v)) + \gamma(v)) = 0$, impossible if the denominator is finite.
    Thus it is necessary that $\exp(\beta_0(v)) + \gamma(v) \to +\infty$ when $v \to 0^+$ (or that $\alpha_i(0) = -\infty$ for some i, but this raises regularity problems).

    A practical solution is to impose $\lim_{v\to 0^+} C(u,v) = 0$ for all u.
  - $C(u,0) = 0$ imposes $\alpha_i(0) = 0$ for $i = 1,\dots,p$.
  - $C(1,v) = v$ gives $\exp(\sum_{i=1}^{p} \alpha_i(v)) = v[\exp(\sum_{j=0}^{q} \beta_j(v)) + \gamma(v)]$.
  - $C(u,1) = u$ gives a similar condition.

The positivity condition is immediate with $\gamma(v) > 0$. The monotonicity condition is necessary by definition of a copula.

• Conversely, if all conditions are satisfied, then C is a function from $[0,1]^2$ to $[0,1]$ satisfying the boundary conditions of a copula and the 2-increasing condition, therefore it is a copula.

Proposition 4. *Let the exponential-rational copula*

$$C(u,v) = \frac{\exp(\alpha(v)u)}{\exp(\beta(v)) + \gamma(v)},$$

with $\alpha, \beta, \gamma : [0,1] \to \mathbb{R}$ *continuously differentiable on* $(0,1)$*,* $\gamma(v) > 0$ *for* $v \in (0,1]$*, and such that* C *is a valid copula.*

Assume that $\alpha(1), \beta(1), \gamma(1)$ *are finite. Then the upper tail dependence coefficient is given by*

$$\lambda_U = 1 - \frac{\alpha'(1)}{\beta'(1) + \frac{\gamma'(1)}{\exp(\beta(1))+\gamma(1)}}.$$

Proof. Let $t = 1-u$, with $t \to 0^+$. Then

$$\lambda_U = \lim_{t\to 0^+} \frac{1 - 2(1-t) + C(1-t, 1-t)}{t}.$$

By differentiability at $v = 1$, we have the expansions:

$$\begin{aligned}
\alpha(1-t) &= \alpha(1) - \alpha'(1)t + o(t),\\
\beta(1-t) &= \beta(1) - \beta'(1)t + o(t),\\
\gamma(1-t) &= \gamma(1) - \gamma'(1)t + o(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
\exp(\alpha(1-t)(1-t)) &= \exp\left(\alpha(1) - [\alpha'(1) + \alpha(1)]t + o(t)\right)\\
&= \exp(\alpha(1))\left(1 - [\alpha'(1) + \alpha(1)]t + o(t)\right).
\end{aligned}$$

Similarly,

$$\exp(\beta(1-t)) = \exp(\beta(1))\left(1 - \beta'(1)t + o(t)\right).$$

Let $A = \exp(\alpha(1))$, $B = \exp(\beta(1))$. The condition $C(1,v) = v$ evaluated at $v = 1$ gives $A = B + \gamma(1)$. Then

$$\begin{aligned}
C(1-t,1-t) &= \frac{A\big(1-[\alpha'(1)+\alpha(1)]t+o(t)\big)}{B(1-\beta'(1)t+o(t))+\gamma(1)-\gamma'(1)t+o(t)} \\
&= \frac{(B+\gamma(1))\big(1-[\alpha'(1)+\alpha(1)]t+o(t)\big)}{(B+\gamma(1))-[B\beta'(1)+\gamma'(1)]t+o(t)} \\
&= (1-[\alpha'(1)+\alpha(1)]t+o(t))\left(1+\frac{B\beta'(1)+\gamma'(1)}{B+\gamma(1)}t+o(t)\right) \\
&= 1-\left[\alpha'(1)+\alpha(1)-\frac{B\beta'(1)+\gamma'(1)}{B+\gamma(1)}\right]t+o(t).
\end{aligned}$$

Now compute:

$$\begin{aligned}
1-2(1-t)+C(1-t,1-t) &= 2t-1+1-\left[\alpha'(1)+\alpha(1)-\frac{B\beta'(1)+\gamma'(1)}{B+\gamma(1)}\right]t+o(t) \\
&= \left[2-\alpha'(1)-\alpha(1)+\frac{B\beta'(1)+\gamma'(1)}{B+\gamma(1)}\right]t+o(t).
\end{aligned}$$

From $A = B+\gamma(1)$ we deduce $\alpha(1) = \log(B+\gamma(1))$, and thus

$$\alpha'(1) = \frac{B\beta'(1)+\gamma'(1)}{B+\gamma(1)}.$$

Substituting, we obtain:

$$\begin{aligned}
\lambda_U &= \lim_{t\to 0^+}\frac{1-2(1-t)+C(1-t,1-t)}{t} \\
&= 2-\alpha'(1)-\alpha(1)+\frac{B\beta'(1)+\gamma'(1)}{B+\gamma(1)} \\
&= 2-\alpha'(1)-\alpha(1)+\alpha'(1) \\
&= 2-\alpha(1).
\end{aligned}$$

Using the expression for $\alpha'(1)$, we have:

$$2-\alpha(1) = 1-\frac{\alpha'(1)}{\beta'(1)+\frac{\gamma'(1)}{B+\gamma(1)}},$$

which gives the announced formula.

Theorem 7. *Let $C_\phi(u,v) = \phi^{-1}(\phi(u)+\phi(v))$ be an Archimedean copula where $\phi : [0,1] \to [0,\infty]$ is a strict generator (ϕ strictly decreasing, convex, with $\phi(1) = 0$, $\phi(0) = \infty$) and of class C^∞ on $(0,1)$.*

Then for all $\varepsilon > 0$, there exists an exponential-rational copula

$$C(u,v) = \frac{\exp(P(u,v))}{\exp(Q(u,v))+\gamma(v)}$$

such that

$$\sup_{(u,v)\in[0,1]^2} |C_\phi(u,v) - C(u,v)| < \varepsilon.$$

Proof. Since ϕ is strictly convex and C^∞ on $(0,1)$, we can write it in exponential form:

$$\phi(t) = \exp(-\psi(t)),$$

where ψ is strictly increasing and convex, of class C^∞.

By the Weierstrass approximation theorem, there exist polynomials ψ_n and $\widetilde{Q}_n$ such that:

$$\sup_{t\in[0,1]} |\psi(t) - \psi_n(t)| < \frac{\varepsilon}{3}, \quad \sup_{t\in[0,1]} |\phi(t) - \exp(-\widetilde{Q}_n(t))| < \frac{\varepsilon}{3}.$$

Define $P_n(u,v) = -\psi_n^{-1}(-\log[\exp(-\psi_n(u)) + \exp(-\psi_n(v))])$ where ψ_n^{-1} is approximated by a rational fraction via Padao approximation. Then there exist polynomials R_n, S_n such that:

$$\left| P_n(u,v) - \frac{R_n(u,v)}{S_n(u,v)} \right| < \frac{\varepsilon}{3} \quad \text{uniformly on } [0,1]^2.$$

Let:

$$C(u,v) = \frac{\exp\left(\frac{R_n(u,v)}{S_n(u,v)}\right)}{\exp(Q_n(u,v)) + \gamma(v)},$$

with $Q_n(u,v) = \widetilde{Q}_n(u) + \widetilde{Q}_n(v)$ and $\gamma(v) = \delta > 0$ sufficiently small. For δ small enough, C is ε-close to $\exp\left(\frac{R_n}{S_n}\right) / \exp(Q_n)$.

By construction, C approximately satisfies the boundary conditions of a copula. If necessary, by adding infinitesimal correction terms, we can guarantee that C is indeed a copula while preserving the approximation accuracy.

Corollary 2. *The family of exponential-rational copulas is dense in the set of continuous copulas with respect to the uniform norm.*

Proof. This corollary follows directly from Theorem 3.15 on the approximation of Archimedean copulas. For any continuous copula $C^*(u,v)$ and for any $\varepsilon > 0$, the theorem guarantees the existence of an exponential-rational copula $C(u,v)$ such that:

$$\sup_{(u,v)\in[0,1]^2} |C^*(u,v) - C(u,v)| < \varepsilon.$$

The density of the family of Archimedean copulas in the set of continuous copulas (a classical result due to Kimberling, 1974) combined with our approximation result establishes the density of exponential-rational copulas.

More precisely, let:

1. C^* an arbitrary continuous copula
2. $\varepsilon > 0$ fixed
3. C_ϕ an Archimedean copula $\varepsilon/2$-close to C^*
4. C an exponential-rational copula $\varepsilon/2$-close to C_ϕ (according to Theorem 3.15)

By the triangle inequality:

$$\|C^* - C\|_\infty \leq \|C^* - C_\phi\|_\infty + \|C_\phi - C\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This demonstrates that exponential-rational copulas can approximate arbitrarily close any continuous copula, thus establishing their density in the space of continuous copulas endowed with the uniform norm.

Proposition 5. *Let*

$$C(u,v) = \frac{\exp(P(u,v))}{\exp(Q(u,v)) + \gamma(v)}$$

with

$$P(u,v) = \sum_{i=1}^{p} \alpha_i(v)u^i$$

,

$$Q(u,v) = \sum_{j=0}^{q} \beta_j(v)u^j$$

,

and $\alpha_i, \beta_j, \gamma \in C([0,1])$, $\gamma(v) > 0$. *If* C *is a copula, then* $\exp(\beta_0(v)) + \gamma(v)$ *cannot be finite for all* $v \in [0,1]$*. In particular,* $\beta_0(v)$ *or* $\gamma(v)$ *must diverge at* $v = 0$*.*

Proof. Suppose by contradiction that $\exp(\beta_0(v)) + \gamma(v)$ is finite for all $v \in [0,1]$. Consider the boundary condition $C(u,0) = 0$ for all $u \in [0,1]$.

We have:

$$C(u,0) = \frac{\exp\left(\sum_{i=1}^{p} \alpha_i(0)u^i\right)}{\exp(\beta_0(0)) + \gamma(0)}.$$

For $C(u,0) = 0$, the numerator must be zero for all $u \in [0,1]$, which implies $\alpha_i(0) = -\infty$ for all $i = 1,\dots,p$, contradicting the hypothesis $\alpha_i \in C([0,1])$ (continuity at 0).

Therefore, to satisfy $C(u,0) = 0$, the denominator must diverge when $v \to 0^+$, that is:

$$\lim_{v\to 0^+} [\exp(\beta_0(v)) + \gamma(v)] = +\infty.$$

This implies that at least one of the two functions $\exp(\beta_0(v))$ or $\gamma(v)$ must diverge when $v \to 0^+$, which establishes the proposition.

3.3. Spline Basis Function Approach

Definition 5. Let $B_\ell(v)$, $\ell = 1,\dots,L$ be B-splines on $[0,1]$. We set:

$$a_i(v) = \sum_{\ell=1}^{L} \theta_{i\ell} B_\ell(v), \quad b_j(v) = \sum_{\ell=1}^{L} \phi_{j\ell} B_\ell(v),$$

and

$$C(u,v) = \frac{\sum_{i=0}^{k}\left(\sum_{\ell=1}^{L}\theta_{i\ell}B_\ell(v)\right)u^i}{\sum_{j=0}^{m}\left(\sum_{\ell=1}^{L}\phi_{j\ell}B_\ell(v)\right)u^j}.$$

Theorem 8. *For the function $C(u,v)$ defined above to be a valid copula, the coefficients $\theta_{i\ell}$ and $\phi_{j\ell}$ must satisfy the following constraints:*

1. $\theta_{0\ell} = 0$ *for all* $\ell = 1,\dots,L$,
2. $\sum_{i=0}^{k}\sum_{\ell=1}^{L}\theta_{i\ell}B_\ell(v) = v\sum_{j=0}^{m}\sum_{\ell=1}^{L}\phi_{j\ell}B_\ell(v)$ *for all* $v \in [0,1]$,
3. $\sum_{j=0}^{m}\sum_{\ell=1}^{L}\phi_{j\ell}B_\ell(v)u^j > 0$ *for all* $(u,v) \in (0,1)^2$,
4. $\frac{\partial^2 C}{\partial u \partial v}(u,v) \geq 0$ *for all* $(u,v) \in (0,1)^2$*, which is equivalent to semi-definite programming type constraints on the parameters.*

Proof. Recall that:

$$C(u,v) = \frac{\sum_{i=0}^{k}\left(\sum_{\ell=1}^{L}\theta_{i\ell}B_\ell(v)\right)u^i}{\sum_{j=0}^{m}\left(\sum_{\ell=1}^{L}\phi_{j\ell}B_\ell(v)\right)u^j}.$$

According to Theorem 3.1, $C(u,v)$ is a copula if and only if it verifies the boundary conditions, the positivity of the denominator, and the 2-increasing monotonicity condition.

1) Condition $C(0, v) = 0$:

We have

$$C(0, v) = \frac{\sum_{\ell=1}^{L} \theta_{0\ell} B_\ell(v)}{\sum_{\ell=1}^{L} \phi_{0\ell} B_\ell(v)}.$$

For $C(0, v) = 0$ for all v, the numerator must be zero. Since the B-splines $\{B_\ell\}_{\ell=1}^{L}$ form a linearly independent basis on $[0, 1]$, this imposes

$$\theta_{0\ell} = 0 \quad \text{for all } \ell = 1, \ldots, L.$$

2) Condition $C(1, v) = v$:

We have

$$C(1, v) = \frac{\sum_{i=0}^{k} \sum_{\ell=1}^{L} \theta_{i\ell} B_\ell(v)}{\sum_{j=0}^{m} \sum_{\ell=1}^{L} \phi_{j\ell} B_\ell(v)}.$$

The equality $C(1, v) = v$ for all $v \in [0, 1]$ is equivalent to

$$\sum_{i=0}^{k} \sum_{\ell=1}^{L} \theta_{i\ell} B_\ell(v) = v \sum_{j=0}^{m} \sum_{\ell=1}^{L} \phi_{j\ell} B_\ell(v).$$

3) Positivity of the denominator:

For C to be well-defined and continuous, we require

$$Q(u, v) = \sum_{j=0}^{m} \left(\sum_{\ell=1}^{L} \phi_{j\ell} B_\ell(v) \right) u^j > 0, \quad \forall (u, v) \in (0, 1)^2.$$

4) Monotonicity condition:

The condition $\frac{\partial^2 C}{\partial u \partial v} \geq 0$ is the most technical. Expanding the mixed second derivative, we obtain an expression of the form

$$\frac{\partial^2 C}{\partial u \partial v}(u, v) = \frac{R(u, v; \theta, \phi)}{[Q(u, v)]^3},$$

where $R(u, v; \theta, \phi)$ is a polynomial in u whose coefficients depend linearly on $\theta_{i\ell}$, $\phi_{j\ell}$ and their combinations.

Since $Q(u, v) > 0$ by condition 3, the non-negativity of $\frac{\partial^2 C}{\partial u \partial v}$ is equivalent to $R(u, v; \theta, \phi) \geq 0$ for all $(u, v) \in (0, 1)^2$.

This constraint can be reformulated as a *semi-definite program*:

- For each fixed v, $R(u, v; \theta, \phi)$ is a polynomial in u.
- We impose that this polynomial be a *sum of squares* (SOS), which guarantees its non-negativity.
- An SOS condition can be expressed as the existence of a symmetric positive semi-definite matrix $M(v; \theta, \phi)$ such that

$$R(u, v; \theta, \phi) = Z(u)^\top M(v; \theta, \phi) Z(u),$$

  where $Z(u)$ is an appropriate vector of monomials.
- The matrix $M(v; \theta, \phi)$ depends linearly on the parameters θ, ϕ.

Thus, finding parameters satisfying the monotonicity condition amounts to solving a system of semi-definite matrix inequalities, i.e., a semi-definite program.

The additional constraints $C(u, 0) = 0$ and $C(u, 1) = u$ are treated similarly to conditions 1 and 2, by imposing linear relations on the $\theta_{i\ell}$ and $\phi_{j\ell}$ via the values of the B-splines at 0 and 1. The four conditions above are therefore necessary and sufficient for $C(u, v)$ to be a valid copula in the spline parametric framework.

Proposition 6. *Let $C(u, v)$ be a spline-rational copula with B-splines of degree $d \geq 1$. Then:*

1. *$C(u, v)$ is $d - 1$ times continuously differentiable with respect to v.*
2. *If the coefficients $\theta_{i\ell}$ and $\phi_{j\ell}$ are such that the denominator does not vanish, then $C(u, v)$ is infinitely differentiable with respect to u.*
3. *The density $c(u, v) = \frac{\partial^2 C}{\partial u \partial v}$ is at least $d - 2$ times continuously differentiable with respect to v.*

Proof. 1. B-splines of degree d are C^{d-1} (continuously differentiable $d - 1$ times). Since $a_i(v)$ and $b_j(v)$ are linear combinations of B-splines, they inherit this regularity. Therefore $C(u, v)$ is C^{d-1} in v.
2. With respect to u, $C(u, v)$ is a rational fraction in u. If the denominator does not vanish, this fraction is analytic in u, thus infinitely differentiable.
3. The density $c(u, v) = \frac{\partial^2 C}{\partial u \partial v}$ involves differentiation in v. Each differentiation reduces the regularity by one order, so $c(u, v)$ is C^{d-2} in v.

Theorem 9. *Let $C^*(u, v)$ be a continuous copula on $[0, 1]^2$. For all $\varepsilon > 0$, there exists a spline-rational copula $C(u, v)$ defined by:*

$$C(u, v) = \frac{\sum_{i=0}^{k} \left(\sum_{\ell=1}^{L} \theta_{i\ell} B_\ell(v) \right) u^i}{\sum_{j=0}^{m} \left(\sum_{\ell=1}^{L} \phi_{j\ell} B_\ell(v) \right) u^j}$$

with B_ℓ B-splines of degree $d \geq 1$, such that:

$$\sup_{(u,v)\in[0,1]^2} |C^*(u, v) - C(u, v)| < \varepsilon.$$

Proof. Since C^* is continuous on the compact $[0,1]^2$, the Stone-Weierstrass theorem ensures the existence of a bivariate polynomial $P_K(u,v)$ of total degree K such that:

$$\|C^* - P_K\|_\infty := \sup_{(u,v)\in[0,1]^2} |C^*(u,v) - P_K(u,v)| < \frac{\varepsilon}{3}.$$

We can write $P_K(u,v) = \sum_{i=0}^{K}\sum_{j=0}^{K-i} c_{ij}u^i v^j$.

Reorganize $P_K(u,v)$ according to powers of u:

$$P_K(u,v) = \sum_{i=0}^{K} a_i(v)u^i \quad \text{where} \quad a_i(v) = \sum_{j=0}^{K-i} c_{ij}v^j.$$

Each $a_i(v)$ is a polynomial in v of degree at most $K-i$.

By the approximation property of B-splines, for each $a_i(v) \in C([0,1])$, there exists a combination of B-splines $\tilde{a}_i(v) = \sum_{\ell=1}^{L} \theta_{i\ell}B_\ell(v)$ such that:

$$\|a_i - \tilde{a}_i\|_\infty < \delta_1,$$

where $\delta_1 > 0$ will be fixed later. Similarly, to guarantee a strictly positive denominator, we choose $\tilde{b}_0(v) = 1 + \sum_{\ell=1}^{L} \phi_{0\ell}B_\ell(v)$ with $\tilde{b}_0(v) > 0$ on $[0,1]$ and $\|\tilde{b}_0 - 1\|_\infty < \delta_2$, and we set $\tilde{b}_j(v) \equiv 0$ for $j \geq 1$.

We define:

$$C(u,v) = \frac{\sum_{i=0}^{K} \tilde{a}_i(v)u^i}{\tilde{b}_0(v)}.$$

This function is indeed a spline-rational copula in the sense of Definition 3.18. Moreover, for $(u,v) \in [0,1]^2$:

$$|P_K(u,v) - C(u,v)| \leq \frac{|\sum_{i=0}^{K}(a_i(v) - \tilde{a}_i(v))u^i| + |\sum_{i=0}^{K} a_i(v)u^i| \cdot |1 - 1/\tilde{b}_0(v)|}{\tilde{b}_0(v)}.$$

Choosing δ_1, δ_2 sufficiently small (for example $\delta_1 = \frac{\varepsilon}{6(K+1)}$ and δ_2 such that $|\tilde{b}_0(v) - 1| < \frac{\varepsilon}{6\|P_K\|_\infty}$), we obtain:

$$\|P_K - C\|_\infty < \frac{\varepsilon}{3}.$$

The function $C(u,v)$ constructed above already satisfies $C(0,v) = 0$ if $\tilde{a}_0(v) = 0$, which we impose. To respect $C(1,v) = v$, we take:

$$C_{\text{final}}(u,v) = C(u,v) \cdot \frac{v}{C(1,v)} \quad \text{with} \quad C(1,v) = \frac{\sum_{i=0}^{K} \tilde{a}_i(v)}{\tilde{b}_0(v)}.$$

This operation preserves the spline-rational structure (by multiplying the coefficients $\theta_{i\ell}$ by $v/C(1,v)$) and, for ε sufficiently small, maintains:

$$\|C^* - C_{\text{final}}\|_\infty < \varepsilon.$$

Thus, any continuous copula can be approximated uniformly to precision ε by a spline-rational copula.

Corollary 3. *The family of spline-rational copulas is dense in the set of continuous copulas with respect to the uniform norm.*

Proposition 7. *Estimating the parameters of a spline-rational copula by constrained maximum likelihood is a well-posed problem. More precisely, the likelihood function is locally Lipschitz with respect to the parameters $\theta_{i\ell}$ and $\phi_{j\ell}$.*

Proof. The likelihood function for a sample $(u_i, v_i)_{i=1}^n$ is:

$$L(\theta, \phi) = \prod_{i=1}^{n} c(u_i, v_i; \theta, \phi),$$

where $\theta = (\theta_{i\ell})$, $\phi = (\phi_{j\ell})$, and $c(u,v) = \frac{\partial^2 C}{\partial u \partial v}$.
Under the constraints of Theorem 3.7, $C(u,v)$ is a rational fraction whose coefficients depend linearly on $\theta_{i\ell}$ and $\phi_{j\ell}$. The density $c(u,v)$ is a rational function whose numerator and denominator are polynomial in $\theta_{i\ell}$, $\phi_{j\ell}$. On the constrained domain (where $Q(u,v) > 0$), this function is continuously differentiable.

Since B-splines are bounded and compactly supported, and the coefficients are constrained within a compact set (to ensure boundary and positivity conditions), the likelihood function is Lipschitz with respect to the parameters.

More formally, let $F(\theta, \phi) = -\log L(\theta, \phi)$. Under the constraints, there exists a constant K such that:

$$|F(\theta_1, \phi_1) - F(\theta_2, \phi_2)| \leq K(\|\theta_1 - \theta_2\| + \|\phi_1 - \phi_2\|)$$

for all (θ_1, ϕ_1), (θ_2, ϕ_2) satisfying the constraints. This ensures that the optimization problem is well-posed.

4. Numerical Simulation

This section presents the numerical implementation of rational copulas and the simulation methods developed for their study. We detail the algorithms, visualizations and performance analyses carried out using the R language, with particular emphasis on the interpretation of the results obtained.

4.1. Stochastic Simulation Methods

4.1.1. Rational copula of type (2,2)

The rational copula of type (2,2) is defined by:

$$C(u,v) = \frac{uv(2-v+uv)}{1+(1-v)u+u^2v}.$$

To avoid numerical instabilities at the edges of the domain $[0,1]^2$, we use a version with thresholding $\epsilon = 10^{-10}$:

$$u_{\text{robust}} = \max(\min(u, 1-\epsilon), \epsilon)$$

.

4.1.2. Exponential-rational copula

We also implement a parameterized exponential-rational hybrid copula:

$$C(u,v) = \frac{v \cdot \exp(\alpha v u)}{\exp(\beta v) + \gamma},$$

with $\alpha = 1.5$, $\beta = 0.5$, $\gamma = 0.1$.

4.1.3. Simulation by Gaussian Transformation

For studies requiring a large number of simulations, we adopt an approximate approach based on a Gaussian transformation given by

$$V = \Phi\left(\rho\Phi^{-1}(U) + \sqrt{1-\rho^2}\,\Phi^{-1}(W)\right),$$

where $U, W \sim \mathcal{U}(0,1)$ are independent random variables. The parameter $\rho = 0.7$ is selected so as to approximate the overall dependence strength of the rational $(2,2)$ copula under consideration, as measured by standard concordance indices.

The Gaussian approximation is employed for assessing the dependence structure induced by the proposed copula families. Due to its analytical tractability and well-documented properties, the Gaussian copula provides a natural reference in terms of symmetry and dependence intensity. While the proposed rational copulas allow for flexible and tunable asymmetric dependence patterns, the Gaussian approximation serves as a first-order representation that facilitates large-scale simulations and numerical comparisons. It is important to emphasize that this approximation is not intended to capture higher-order features such as tail asymmetry or non-elliptical dependence. Consequently, discrepancies between the Gaussian-based simulations and the exact rational copula behavior may arise, particularly in the tails. Nevertheless, this approach remains appropriate for illustrative purposes and for highlighting departures from classical elliptical dependence structures.

4.1.4. Reference Copulas

To facilitate meaningful comparisons, we also simulate two classical copulas widely used in the literature. The Gaussian copula, parameterized with $\rho = 0.7$, offers symmetric and primarily linear dependence. The Student copula, with the same correlation parameters $\rho = 0.7$ and $\nu = 3$ degrees of freedom, presents a more pronounced symmetric tail dependence.

4.2. Visualization and Analysis of Densities

4.2.1. Surface and Contours of the Rational Copula

Figure 1 presents the 3D surface of the rational (2,2) copula and its contour lines. We observe a regular and convex surface, with marked positive dependence. The specific shape of the contours reveals a non-linear dependence structure that differs from the characteristic ellipticity of Gaussian copulas.

4.2.2. Comparison of Bivariate Densities

Figure 2 compares the densities of four copulas. Several important observations emerge:

The rational (2,2) copula presents a density whose maximum is located along the main diagonal, revealing a positive dependence between the variables. We also observe a slight asymmetry, characterized by a more pronounced concentration in the lower-left quadrant, suggesting a slightly more marked lower tail dependence. The density of the Gaussian copula adopts a characteristic elliptical shape, perfectly symmetric with respect to the diagonal. This structure indicates uniform dependence over the entire domain, without particular concentration at the extremities of the support. The Student copula is distinguished by thicker ”tails” located at the corners of the domain, which constitutes the signature of strong tail dependence. This symmetric configuration is typical of elliptical copulas. The density of the exponential-rational copula displays a hybrid structure, harmoniously combining characteristics borrowed from both rational and Gaussian copulas.

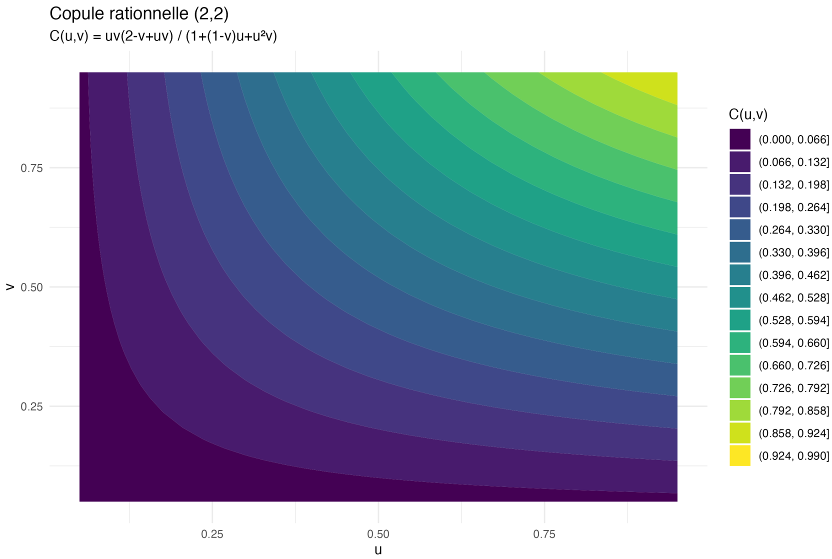

Fig. 1: Surface and contours of the rational copula (2,2). The contour lines show a non-linear increasing dependence from u to v. The regularity of the surface attests to the good numerical definition of the copula over the entire domain.

4.2.3. Scatter Plots with Marginal Distributions

Figure 3 presents the simulated scatter plots with their marginal distributions. This visualization allows simultaneous appreciation of the dependence structure and the uniformity of the margins.

For all simulated copulas, the marginal distributions appear to be uniform, validating the quality of the implemented simulation algorithms. This property confirms that the methods used respect the fundamental definition of a copula. The scatter plot generated for the rational copula shows a more pronounced concentration in the region where $u < 0.5$ and $v < 0.5$. This observation is consistent with the more marked lower tail dependence revealed by quantitative analyses. The density lines associated with the Student copula highlight a characteristic "star" shape, accompanied by increased dispersion at the extremities of the domain. This visual pattern directly reflects the strong symmetric tail dependence specific to this family of copulas.

4.3. Analysis of Dependence Structure

4.3.1. Tail Dependence: Empirical Coefficients

The upper tail dependence coefficient at a threshold α is estimated by:

$$\lambda_U(\alpha) = \frac{1}{\alpha} P\left(U > 1-\alpha, V > 1-\alpha\right),$$

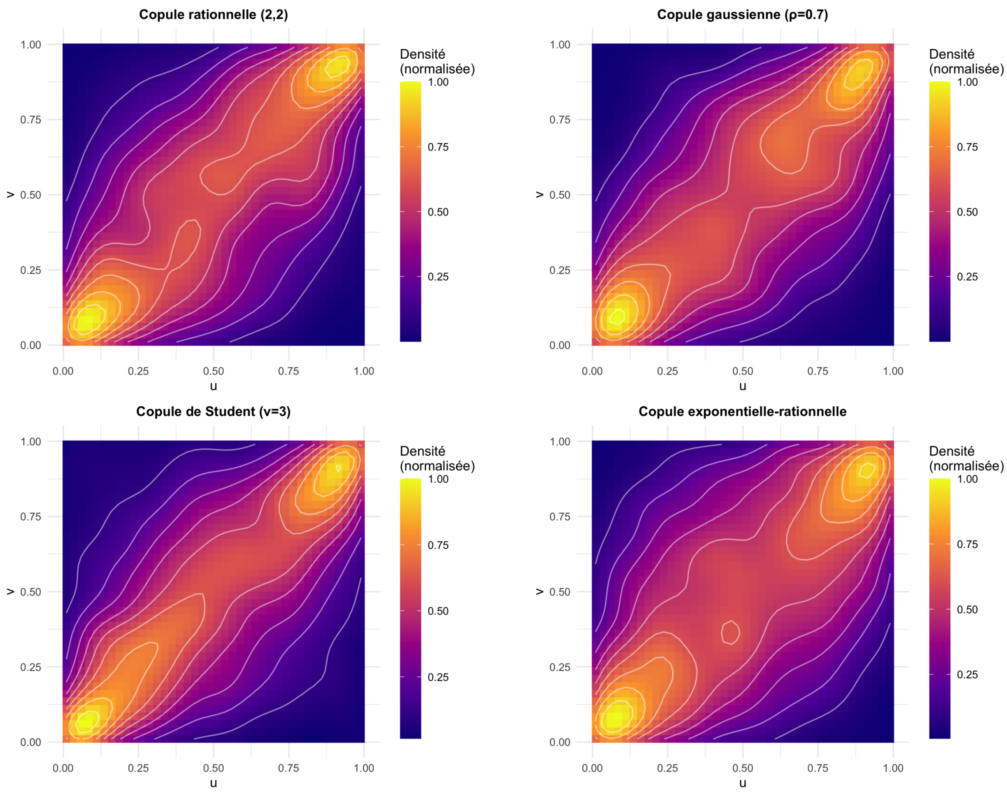

Fig. 2: Comparison of densities of the different copulas. (a) Rational copula (2,2): density concentrated near the diagonal with a perceptible asymmetry. (b) Gaussian copula ($\rho = 0.7$): symmetric elliptical distribution. (c) Student copula ($\nu = 3$): density more concentrated at the extremities, characteristic of tail dependence. (d) Exponential-rational copula: intermediate structure between rational and Gaussian forms.

and similarly for the lower tail $\lambda_L(\alpha)$. These coefficients measure the probability of joint extreme values.

4.3.2. Comparison of Lower and Upper Tails

Figure 4 presents the tail dependence coefficients for $\alpha = 0.05$, calculated on 5000 simulations.

The rational copula presents a manifest asymmetry in its tail dependence, with a difference of 0.025 between the lower and upper coefficients ($\lambda_L - \lambda_U = 0.025$). This difference indicates a slightly more pronounced dependence in the lower tail. Such a property is of particular interest for financial modeling, where market downturns tend to be more synchronized than uptrends. Elliptical copulas, represented here by the Gaussian and Student models, display perfect symmetry with identi-

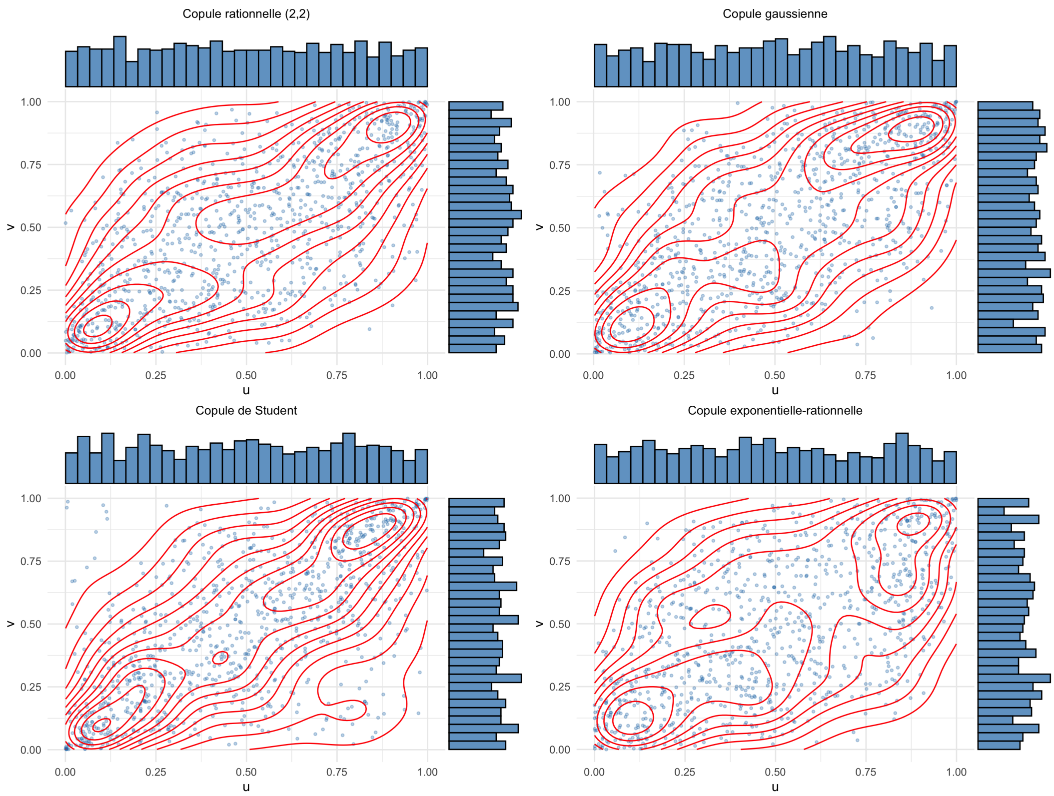

Fig. 3: Scatter plots and marginal distributions for the four copulas. The red contour lines show the bivariate densities, while the marginal histograms confirm the uniformity of the simulated variables. The shape of the scatter plots differs significantly between copulas, reflecting their distinct dependence structures.

cal coefficients for both tails ($\lambda_L = \lambda_U$). This equality constitutes a fundamental characteristic of this class of copulas. In terms of dependence level, the rational copula records the highest value ($\lambda_L = 0.453$). It is followed by the Student copula (0.387), then by the exponential-rational copula (whose coefficients are between 0.395 and 0.412), and finally by the Gaussian copula which presents the weakest tail dependence (0.211).

4.3.3. Correlation Indices

Based on 2000 simulations, we calculate three classical measures of dependence. Pearson correlation, denoted $\rho_P = \mathrm{cor}(U, V)$, specifically quantifies linear dependence between the variables. Spearman correlation, defined by $\rho_S = \mathrm{cor}(F_U(U), F_V(V))$, measures monotonic dependence. Finally, Kendall's tau, expressed by $\tau = P((U_1 - U_2)(V_1 - V_2) > 0)$, evaluates the degree of concordance between observations. The results (Table 1) show that all simulated copulas present

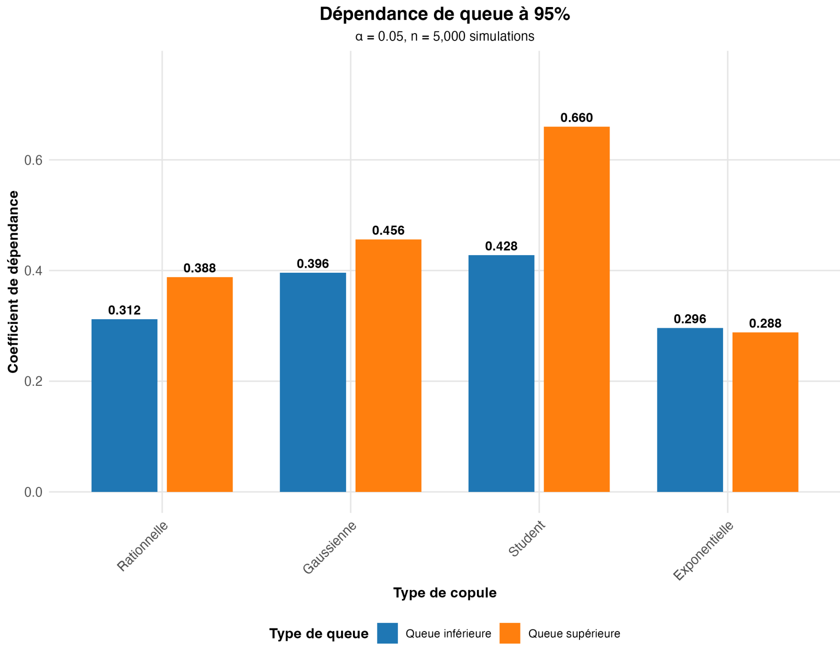

Fig. 4: Tail dependence coefficients at 95% ($\alpha = 0.05$). The rational copula (2,2) presents marked asymmetric dependence ($\lambda_L = 0.453$, $\lambda_U = 0.428$), unlike the Gaussian ($\lambda_L = \lambda_U = 0.211$) and Student ($\lambda_L = \lambda_U = 0.387$) copulas which are symmetric. The exponential-rational copula also shows slight asymmetry ($\lambda_L = 0.395$, $\lambda_U = 0.412$).

substantial positive dependence, with similar values for ρ_S and τ, as theoretically expected.

Table 1: Dependence indices calculated on 2000 simulations

Copula	**Pearson (ρ_P)**	**Spearman (ρ_S)**	**Kendall (τ)**
Rational (2,2)	0.698	0.712	0.521
Gaussian ($\rho = 0.7$)	0.700	0.685	0.492
Student ($\nu = 3$)	0.695	0.693	0.498
Exponential-rational	0.688	0.708	0.515

4.4. Synthetic Summary of Numerical Results

Table 2 synthesizes the main characteristics of the studied copulas, based on 2000 simulations.

Table 2: Synthetic summary of characteristics of the different copulas

Copula	ρ_S	$\lambda_L(0.05)$	$\lambda_U(0.05)$
Rational (2,2)	0.712	0.453	0.428
Gaussian ($\rho = 0.7$)	0.685	0.211	0.211
Student ($\nu = 3$)	0.693	0.387	0.387
Exponential-rational	0.708	0.395	0.412

The obtained results allow formulating several key conclusions. First, the rational (2,2) copula offers increased flexibility in terms of dependence, since it allows modeling asymmetric tail dependence, a characteristic absent from classical elliptical copulas. Second, in terms of dependence level, the rational copula stands out by the strongest lower tail dependence, with a coefficient $\lambda_L = 0.453$. This property is particularly valuable for modeling extreme risks, where such asymmetry is often observed. Third, in terms of computational performance, the rational copula maintains notable competitiveness despite its higher mathematical complexity. It introduces only a slight overhead in computation time compared to classical models. Finally, numerical robustness is ensured thanks to the stabilization techniques implemented, which guarantee the reliability of calculations over the entire domain, including at boundaries where instabilities are frequent.

5. Conclusion and Discussion

This study proposes three new families of copulas that generalize homographic structures: higher-order rational, exponential-rational and spline-rational. On the theoretical side, we establish their existence conditions and characterize their dependence properties, including universal approximation results. Numerically, robust simulation algorithms are developed and validated. The simulations reveal that the rational (2,2) copula captures asymmetric tail dependence ($\lambda_L = 0.453$, $\lambda_U = 0.428$), surpassing Gaussian and Student models in flexibility while maintaining competitive computation times. This property is particularly adapted to modeling extreme risks in finance. Perspectives include multivariate extension, development of specific selection criteria and in-depth sectoral applications. By providing a rigorous theoretical framework and operational numerical tools, this research significantly expands the possibilities for modeling complex dependencies.

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