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Almost sure convergence of partitioning regression estimates under random right censoring

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Abstract. We study non parametric partitioning regression estimates based on right censored data of the Gneyou(2005) type, associated with a probability weight function. We establish their almost sure uniform convergence over a family of measurable functions on $\mathbb{R}_+$, indexed by a parameter $s \in S$. This result extends the work of Carbonez, Györfi, and Van der Meulen (1995), as well as that of Kohler, Máthé, and Pintér (2002).

Key words: Censored data; nonparametric partitioning; regression estimates; almost sure uniform convergence, measurable functions
AMS 2010 Mathematics Subject Classification Objects : 62GL05; 62G08; 62N01.

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Résumé. (Abstract in French) Nous étudions des estimateurs non paramétriques de régression par partitionnement, basés sur des données censurées à droite du type Gneyou (2005), associés à une fonction de pondération probabiliste. Nous établissons leur convergence uniforme presque sûre sur une famille de fonctions mesurables définies sur $\mathbb{R}_+$, indexée par un paramètre $s \in S$. Ce résultat étend les travaux de Carbonez, Györfi et Van der Meulen (1995), ainsi que ceux de Kohler, Máthé et Pintér (2002). .

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1. Introduction

Let Z be a random vector taking values in $\mathbb{R}^d$, and let X be a square-integrable real-valued random variable. The components of Z may follow different types of distributions, some discrete (e.g., binary), and others absolutely continuous. In regression analysis, the goal is to estimate the variable X based on the observation of Z. In other words, one seeks to determine a function φ defined on $\mathbb{R}^d$ such that $\varphi(Z)$ is as close as possible to X. The main objective of this approach is to minimize the mean squared error

$$\min_{\varphi} \mathbb{E}\left[(\varphi(Z) - X)^2\right] \tag{1}$$

As well established in statistics, this minimum is attained by the regression function $r(z) = \mathbb{E}[X \mid Z = z]$. This function represents the best approximation of X by a function of Z, in the sense of minimizing the mean squared error. Its generalized form is given by:

$$r_\psi(z) = \mathbb{E}[\psi(X) \mid Z = z] \tag{2}$$

where ψ belongs to a family of measurable real-valued functions on $\mathbb{R}_+$. For every measurable function φ, we have

$$\begin{aligned}\mathbb{E}\left[(\varphi(Z) - \psi(X))^2\right] &= \mathbb{E}\left[(r_\psi(z) - \psi(X))^2\right] + \mathbb{E}\left[(r_\psi(z) - \varphi(Z))^2\right]\\ &= \mathbb{E}\left[(r_\psi(z) - \psi(X))^2\right] + \int \mid r_\psi(z) - \varphi(Z) \mid^2 \mu(dz)\end{aligned}$$

where ψ belongs to a family of measurable real-valued functions on $\mathbb{R}_+$. For every measurable function φ, we have

$$\mathbb{E}\left[(\varphi(Z)-\psi(X))^2\right]=\mathbb{E}\left[(r_\psi(z)-\psi(X))^2\right]+\mathbb{E}\left[(r_\psi(z)-\varphi(Z))^2\right]$$
$$=\mathbb{E}\left[(r_\psi(z)-\psi(X))^2\right]+\int \mid r_\psi(z)-\varphi(Z)\mid^2 \mu(dz)$$

where μ denotes the distribution of Z. Following Prakasa Rao Prakasa (1983), we introduce the integrated squared (or excess) error of the function φ, defined by

$$J(\varphi)=\int \mid r_\psi(z)-\varphi(Z)\mid^2 \mu(dz) \tag{3}$$

Note that the mean squared error of φ is close to its minimum if and only if the excess (or integrated squared) error tends to zero. When the joint distribution of (X, Z) is known, the regression function can, at least in theory, be computed explicitly from its definition, in which case the excess error is zero.

In practice, we consider a sample of size n, consisting of pairs of i.i.d. random variables (X_i, Z_i), and denote by (X, Z) a generic instance of such a pair. The underlying joint distribution $F(x,z)$, the joint density $f(x,z)$, and the conditional distribution $F(x \mid z)$ are all unknown. Let $C_1, C_2, ..., C_n$ be another sequence of i.i.d. real-valued random variables with the same distribution as a random variable C, having cumulative distribution function G, and assumed independent of the (X_i, Z_i). The random variables C_i, X_i , and Z_i are all defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We aim to estimate the regression function 2 in such a way that the excess error vanishes asymptotically. Unfortunately, in many medical studies, the variables X_i are not always directly observable. If X represents a patient's survival time, three main situations may occur: some patients are still alive at the end of the study, some withdraw from the study while still alive, and others die from causes unrelated to the condition under investigation. In such cases, the actual survival time is generally not directly observed. Instead, we observe the minimum between the survival time X and a censoring time C, so that the available data take the following form:

$$(Y_i, \delta_i, Z_i)_{1\le i\le n}\, o\grave{u}\, Y_i=\min(X_i, C_i), \delta_i=\mathbb{I}_{\{X_i\le C_i\}}=\begin{cases}1 & if \quad X_i \le C_i\\ 0 & else\end{cases}$$

where $\mathbb{I}_A$ denotes the indicator function of set A.

We aim here to develop a nonparametric estimation method for the regression function $r_\psi(z)$, based on censored data of the form $(Y_i, \delta_i, Z_i)_{1\le i\le n}$, in the case where the covariate space is subject to a finite or countably infinite measurable partition. To ensure the almost sure existence of $r_\psi(z)$, we will assume throughout that $\mathbb{E}[\mid \psi(X)\mid] < +\infty$.

Partitioning estimation has been the subject of various studies in the literature, both for censored and uncensored data. Tukey (1947), Tukey (1961a) was the first to introduce the concept of partitioning, which was later explored in more depth by Collomb (1977), as well as by Bosq (1987). Devroye*et al.* (1983) established, without assuming any specific distribution for the data, an exponential bound on the L_1 error of estimators based on regular partitions of the covariate space. Devroye (1996) demonstrated the universal consistency properties of partition-based estimators. Beirlant and Györfi (1998) provided a sophisticated asymptotic analysis of the L_2 error for partitioning estimators, showing not only their mean bias but also the asymptotic normality of their random fluctuations, while Györfi *et al.* (2002) demonstrated the relative stability of the estimator. In the presence of random censoring, Carbonez *et al.*(1995) established strong asymptotic properties (consistency, uniform convergence) for partitioning estimators. Kohler *et al.* (2002) considered partition based regression estimators under right censored data, using a local averaging method. They showed that the mean squared error converges almost surely to zero regardless of the distribution of the triplet (X, Z, C).

Our work improves upon the results of Carbonez *et al.*(1995) and Kohler *et al.* (2002) by relaxing restrictive assumptions such as strong regularity and absolute continuity, thus providing a more general framework. In addition, we explicitly address random right censoring, which is crucial in survival analysis but not central in earlier works. Finally, through our simulation study (see script), we illustrate the convergence of the proposed estimator. Without imposing any absolute continuity condition on the distribution of Z, nor any regularity (smoothing) assumptions on the regression function $r_{\psi}(z)$, we develop a histogram-type approach based on partitioning estimation, adapted to censored data of the form $(Y_i, \delta_i, Z_i)_{1\leq i\leq n}$. Our analysis focuses on the almost sure convergence of the estimator of $r_{\psi_s}(z)$, where ψ_s, indexed by $s \in S \subset \mathbb{R}$, generalizes the work of Carbonez *et al.*(1995), as well as that of Kohler *et al.* (2002) thereby enabling the estimation of the conditional distribution function.

This article is structured as follows. Section 2 introduces the definitions of the estimators along with the necessary assumptions. Section 3 is devoted to the study of the almost sure convergence of the estimators. Section 4 presents the simulation study, and section 5 presents the proof of the main theorem, while section 6 concludes the article.

2. Estimators and Assumptions

We assume that the covariate vector Z is available and used to predict $\psi_s(X)$. The value of the regression function at a point z then corresponds to the conditional expectation $\mathbb{E}[\psi_s(X) \mid Z = z]$, for $s \in S$. This regression function can be estimated using local averaging, which consists in aggregating the values $\psi_s(X_i)$ corresponding to the observations Z_i that lie near z. An explicit form of this estimator is given by :

$$r_{\psi_s,n}(z)=\sum_{i=1}^{n}\mathscr{W}_{n,i}(z)\,\psi_s(X_i) \tag{4}$$

where ψ_s, for $s\in S$, belongs to a family $\mathcal{F}$ of measurable functions on $\mathbb{R}_+$ satisfying the following conditions:
$(F1)$ $\forall s<s'\in S,\ 0\le\psi_s(x)\le\psi_{s'}(x),\forall x\in\mathbb{R}$
$(F2)$ The family $\mathcal{F}$ admits a measurable and bounded envelope function $\Psi(x)=\sup_{s\in S}\mid\psi_s(x)\mid$ with finite first and second moments.

The weight functions $\mathscr{W}_{n,i}(z)=\mathscr{W}_{n,i}(z,Z_1,Z_2,...,Z_n)$ depend both on the evaluation point z and on the entire set of covariates $Z_1,Z_2,Z_3,...,Z_n$. We consider a partition $\mathscr{P}_n=\{A_{n,1},A_{n,2},...,\}$ which is either finite or countably infinite and measurable, covering $\mathbb{R}^d$. For any $z\in\mathbb{R}^d$, let $A_n(z)$ denote the unique element of the partition $\mathscr{P}_n$ that contains z. The weight function induced by the partition $\mathscr{P}_n$ is defined by

$$\mathscr{W}_{n,i}(z)=\begin{cases}\sum_j \frac{\mathbb{I}_{\{Z_i\in A_{n,j}(z)\}}}{\sum_{k=1}^{n}\mathbb{I}_{\{Z_k\in A_{n,j}(z)\}}}\mathbb{I}_{\{z\in A_{n,j}(z)\}} & if\ \sum_{k=1}^{n}\mathbb{I}_{\{Z_k\in A_{n,j}(z)\}}>0\\ 0 & else\end{cases} \tag{5}$$

and satisfying the following conditions:

(i) There exists a number $\eta\ge1$ such that

$$\mathbb{P}\left\{\sum_{i=1}^{n}\frac{\mathscr{W}_{n,i}^2(z)}{\mathbb{E}\left[\sum_{i=1}^{n}\mathscr{W}_{n,i}^2(z)\right]}\le\eta\right\}=1$$

(ii) There exists a number $D\ge1$ such that

$$\mathbb{P}\left\{\sum_{i=1}^{n}\mid\mathscr{W}_{n,i}(z)\mid\le D\right\}=1$$

(iii) The weights $\mathscr{W}_{n,i}(z)$ are decreasing with respect to i and that $\forall\beta>1,\mathbb{E}\left[\sum\mathscr{W}_{n,i}^2(z)\right]\le\frac{C'(z)}{i^\beta}:=C''(z)$ and there exist $\alpha>d$ such that $C''(z)\underset{\|z\|\to\infty}{=}\mathcal{O}(\|z\|^{-\alpha})$

(iv)

$$\lim_{n\to+\infty}\left\{\mathbb{E}\left[\sum\mathscr{W}_{n,i}^2(z)\right]\right\}=0$$

(v) $\forall\epsilon>0$,

$$\lim_{n\to+\infty}\mathbb{P}\left[\mid\sum_{i=1}^{n}\mathscr{W}_{n,i}(z)-1\mid>\epsilon\right]=0$$

In the context of the random right censoring model, the partitioning estimator of the regression function,when only the observations $\{(Y_i,\delta_i,Z_i)\}_{1\le i\le n}$are available, is given by the following expression:

$$\overline{r}_{\psi_s,n}(z)=\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i\psi_s(Y_i)}{1-G(Y_i)}\mathscr{W}_{n,i}(z), s\in S \tag{6}$$

In the absence of covariates, the unbiased estimator of the regression function is given by the following expression:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i\psi_s(Y_i)}{1-G(Y_i)} \tag{7}$$

Indeed

$$\begin{aligned}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i\psi_s(Y_i)}{1-G(Y_i)}\right] &= \mathbb{E}\left[\frac{\mathbb{I}_{\{X_1\le C_1\}}\psi_s(X_1)}{1-G(X_1)}\right]\\ &=\mathbb{E}\left\{\frac{\psi_s(X_1)}{1-G(X_1)}\mathbb{E}\left[\mathbb{I}_{\{X_1\le C_1|X_1\}}\right]\right\}\\ &=\mathbb{E}\left[\psi_s(X_1)\mid X_1\right]\\ &=\mathbb{E}\left[\psi_s(X)\right]\end{aligned}$$

Let

$$\begin{aligned}\tau_F &= \sup\{x\in\mathbb{R}: 1-F_X(x)=\mathbb{P}[X>x]>0\}\\ \tau_G &= \sup\{x\in\mathbb{R}: 1-G(x)>0\}\\ \tau_K &= \tau_F\wedge\tau_G\end{aligned}$$

In practice, τ_F typically denotes the upper limit of the observation period for X. We make the following assumptions:
$(F3)$ G is continuous and $\exists\tau<\tau_F<\infty$, such that $1-G(\tau)>0$.

In survival analysis, the distribution function of the censoring variable G is generally unknown. We therefore replace it with its empirical estimator, denoted G_n, which corresponds to the Kaplan–Meier estimator Kaplan and Meier (1958) of G, defined as follows:

$$1-G_n(t)=\prod_{\substack{i:X_{i,n}\le t\\ 1\le i\le n}}\left(1-\frac{1-\delta_{i,n}}{n-i+1}\right) \tag{8}$$

$X_{1;n}\le X_{2,n}\le ...\le X_{n,n}\coloneqq\tau_{K,n}$, where X_j denotes the ordered values and $\delta_{j,n}$ the corresponding δ_j values associated with them $X_{i,n}=X_j, 1\le j\le n$(see e.g. Deheuvels and Einmhal (2000)).

The estimator defined in 8 satisfies, for all $t\in\mathbb{R}$:
$(F4)$ $\sup_{t\le\tau_K}\mid\frac{1-G_n(t)}{1-G(t)}-1\mid\to 0\,p.s.$ when $n\to+\infty$
$(F5)$ $\tau_{K,n}$ converges almost surely to τ_K.

Thus, we obtain a nonparametric partitioning estimator of $r_{\psi_s,}(z)$ of the form:

$$\widetilde{r}_{\psi_s,n}(z) = \frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i\psi_s(Y_i)}{1-G_n(Y_i)}\mathscr{W}_{n,i}(z), s\in S \tag{9}$$

When $r_{\psi_s,}(z)$ is associated with a Kernel K, we obtain the Kaplan-Meier kernel estimator of the regression function defined by

$$\check{r}_{\psi_s,n}(z) = \frac{1}{nh_n}\sum_{i=1}^{n}\frac{\delta_i\psi_s(Y_i)}{1-G_n(Y_i)}\frac{K\left(\frac{z-Z_i}{h_n}\right)}{\sum_{i=1}^{n}K\left(\frac{z-Z_i}{h_n}\right)} \tag{10}$$

Where h_n is a smoothing parametric(For further details, see Gneyou (2005)).
When $\psi_s(x) = \mathbb{I}_{\{x\leq s\}}$, we obtain an estimator $\widetilde{F}_n(s \mid z)$ of the conditional distribution function $F(s\mid z) = \mathbb{P}[X\leq s \mid Z = z]$, given by:

$$\widetilde{F}_n(s\mid z) = \frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i\mathbb{I}_{\{Y_i\leq s\}}}{1-G_n(Y_i)}\mathscr{W}_{n,i}(z), s\in S \tag{11}$$

3. Main results

Our result concerns the strong consistency of the estimator 9.

Theorem 1. *Let $\widetilde{r}_{\psi_s,n}, s\in S$ be associated with a positive weight function $\mathscr{W}$ such that assumptions $(i-v)$ and $(F1-F5)$ are satisfied. Then, for sufficiently large n,*

$$\int_{\mathbb{R}^d}\sup_{s\in S}\mid \widetilde{r}_{\psi_s,n}(z) - r_{\psi_s}(z)\mid^2 \mu(dz) = o(1)\ p.s. \tag{12}$$

for all distribution of (X, Z, C).

Corollary 1. *Let $\widetilde{F}_n(s\mid z)$ be the estimator of the conditional distribution function $F(s\mid z)$associated with a weight function $\mathscr{W}$ such that assumptions $(i)-(v)$ and $(F3)-(F5)$ are satisfied. Then, for sufficiently large n*

$$\sup_{s\leq\tau_F}\mid \widetilde{\widetilde{F_n}}(s\mid z) - F(s\mid z)\mid = o(1)\ p.s. \tag{13}$$

for all distribution of (X, Z, C).

4. Simulation study

In this section, without loss of generality, we restrict our attention to the case $d = 1$ Simulation studies are performed to evaluate the estimator's performance against the theoretical curve. We conduct simulation studies based on generated data. The simulated samples are obtained via the inversion method: the covariate Z is first drawn from a uniform distribution over the interval $[0; 5]$, after which X is generated conditionally on Z. Censoring times are produced from an exponential distribution with a rate corresponding to 20%. The sample size is denoted by n. As we said in the introduction, taking a family of functions $\{\psi_s; s \in \mathbb{R}_+\}$ such that $\psi_s(x) = \mathbb{I}_{\{x \leq s\}}$ we have $r_{\psi_s}(z) = \mathbb{E}\left[\mathbb{I}_{\{x \leq s\}} \mid Z = z\right] = \mathbb{P}[X \leq s \mid Z = z] = F(s \mid z)$. Then, it follows by 11 that

$$\frac{1}{n} \sum_{i=1}^{n} \frac{\delta_i \mathbb{I}_{\{Y_i \leq s\}}}{1 - G_n(Y_i)} \mathscr{W}_{n,i}(z), s \in \mathbb{R}_+$$

is a consistent estimator of $F(s \mid z)$. The following figures display the estimates of $F(s \mid z)$ for $z \in \{0; 1, 2; 3; 4, 5\}$, obtained with different sample sizes $n \in \{100; 500; 1000; 10000; 100000; 150000\}$. We can check that the the estimator behaves well for large sample size.

It is also interesting to see in the following table that our estimators are consistent and that both the bias and the standard deviations of our estimators tend to 0 as the sample size increases.

z	0	1	2	3	4	5
$F(s \mid z)$	0.0228	0.3933	0.4854	0.4980	0.4997	0.5000
	0.1612	0.3768	0.4936	0.5034	0.5005	0.4971
$n = 100$	0.1384	-0.0165	0.0082	0.0054	0.0008	-0.0029
	0.1422	0.1842	0.1900	0.1852	0.1864	0.1733
	0.1467	0.3740	0.4927	0.4952	0.4995	0.4965
$n = 500$	0.1240	-0.0193	0.0073	-0.0029	-0.0003	-0.0034
	0.0587	0.0790	0.0804	0.0791	0.0740	0.0748
	0.1456	0.3733	0.4897	0.4989	0.5017	0.4993
$n = 1000$	0.1229	-0.0200	0.0043	0.0009	0.0020	-0.0007
	0.0411	0.0556	0.0563	0.0555	0.0563	0.0557

Table 1. Estimation of $F(s \mid z)$ for $z \in \{0, 1, 2, 3, 4, 5\}$: mean, bias, and standard deviation. Each cell reports (mean, bias, standard deviation) of $R = 1000$ estimates for various sample sizes n.

Remark 1. The discrepancy between the estimated curve and the true curve in the region $z \in [0; 1, 250]$ can be attributed to several factors acting jointly. First, the relatively small number of observations in this interval reduces estimation accuracy. Second, the regression function $F(s \mid z)$ varies rapidly in this range, which increases the partitioning bias since piecewise-constant approximations become less accurate. Finally, censoring effects amplify the variability of the therm $\frac{\delta_i}{1-G_n(.)}$, further contributing to the instability of the estimator.

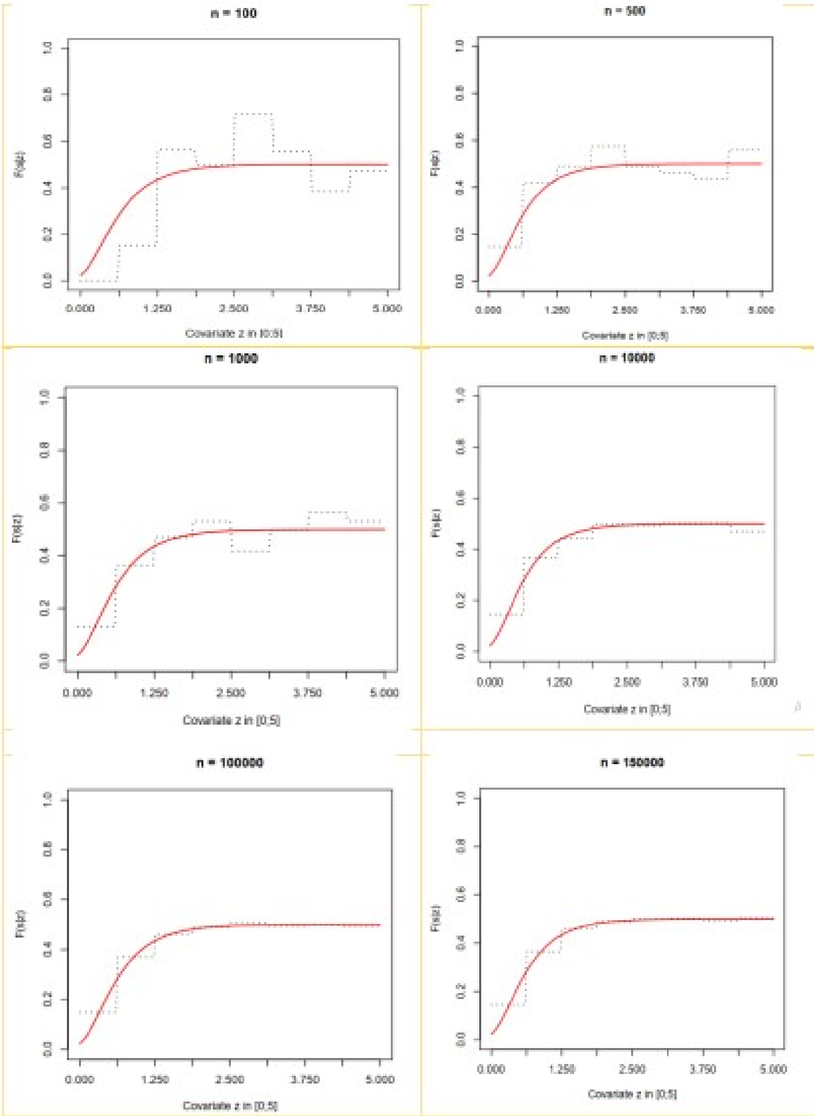

Fig. 1. Comparison of $F\,(s \mid z)$(solid line) and its estimates (dashed line) for various values of n.

5. Proofs of theorem 1 and corollary 1

5.1. Proof of theorem 1

Proof. By applying the triangle inequality, we obtain::

$$
\begin{aligned}
\mid \widetilde{r}_{\psi_s,n}(z) - r_{\psi_s}(z) \mid \leq & \mid \widetilde{r}_{\psi_s}(z) - \overline{r}_{\psi_s,n}(z) \mid + \mid \overline{r}_{\psi_s,n}(z) - \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right] \mid \\
& + \mid \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right] - \overline{r}_{\psi_s,n}(z) \mid \\
\mid \widetilde{r}_{\psi_s,n}(z) - r_{\psi_s}(z) \mid^2 \leq & \, 3 \mid \widetilde{r}_{\psi_s}(z) - \overline{r}_{\psi_s,n}(z) \mid^2 + 3 \mid \overline{r}_{\psi_s,n}(z) - \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right] \mid^2 \\
& + 3 \mid \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right] - r_{\psi_s}(z) \mid^2
\end{aligned}
$$

Hence

$$
\int_{\mathbb{R}^d} \sup_{s\in S} \mid \widetilde{r}_{\psi_s,n}(z) - r_{\psi_s}(z) \mid^2 \mu(dz) \leq 3\left(\mathscr{R}_n(z) + \mathscr{P}_n(z) + \mathscr{Q}_n(z)\right)
$$

where

$$
\begin{aligned}
\mathscr{R}_n(z) &= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \widetilde{r}_{\psi_s}(z) - \overline{r}_{\psi_s,n}(z) \mid^2 \mu(dz) \\
\mathscr{P}_n(z) &= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \overline{r}_{\psi_s,n}(z) - \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right] \mid^2 \mu(dz) \\
\mathscr{Q}_n(z) &= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right] - r_{\psi_s}(z) \mid^2 \mu(dz)
\end{aligned}
$$

To establish result 12, its suffices to show that each of terms $\mathscr{R}_n(z), \mathscr{P}_n(z)$ and $\mathscr{Q}_n(z)$ converges almost surely to zero. We have the following lemmas below. ■

Lemma 1. *If the assumptions* (i),(iii),(iv) *et* $(F1 - F5)$ *are satisfied. Then, for sufficiently large* n,

$$
\mathscr{R}_n(z) = o(1)\ p.s. \tag{14}
$$

Proof. We have the following equality

$$\begin{aligned}
\mathscr{R}_n(z) \leq & \int_{\mathbb{R}^d} \frac{\left(\sup_{s\in S} \mid \psi_s(\tau) \mid\right)^2}{n\left[1-G(\tau)\right]^2} \\
& \times \sum_{i=1}^{n} n\mathscr{W}_{n,i}^2(z) \sup_{t\leq\tau_F} \mid G_n(t)-G(t)\mid^2 \mu(dz) \\
\leq & \left[\frac{\sup_{s\in S} \mid \psi_s(\tau)\mid}{1-G(\tau)}\right] \sup_{t\leq\tau_F} \mid G_n(t)-G(t)\mid^{2\ 2} \\
& \times \int_{\mathbb{R}^d} \mathbb{E}\left[\sum_{i=1}^{n} \mathscr{W}_{n,i}^2(z)\right] \sum_{i=1}^{n} \frac{\mathscr{W}_{n,i}^2(z)}{\mathbb{E}\left[\sum_{i=1}^{n} \mathscr{W}_{n,i}^2(z)\right]} \mu(dz) \\
\leq & \eta \left[\frac{\sup_{s\in S} \mid \psi_s(\tau)\mid}{1-G(\tau)}\right]^2 \\
& \times\left(\sup_{t\leq\tau_F} \mid G_n(t)-G(t)\mid\right)^2 \int_{\mathbb{R}^d} \mathbb{E}\left[\sum_{i=1}^{n} \mathscr{W}_{n,i}^2(z)\right] \mu(dz)
\end{aligned}$$

Since $\mathbb{E}\left[\sum_{i=1}^{n} \mathscr{W}_{n,i}^2(z)\right] \to 0$ as $n \to +\infty$ almost everywhere on $\mathbb{R}^d$ and since under assumption (iii) there exists an integrable function $C'' : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}\left[\sum_{i=1}^{n} \mathscr{W}_{n,i}^2(z)\right] \leq C''(z)$, it follows from the dominated convergence theorem and assumption $(F4)$ that result 14 holds. ■

Lemma 2. *If the assumptions* $(ii),(v)$ *et* $(F1-F5)$ *are satisfied. Then, for sufficiently large* n*,*

$$\mathscr{P}_n(z) = o(1)\ p.s. \tag{15}$$

Proof. We have

$$\begin{aligned}
\mid \overline{r}_{\psi_s,n}(z) - \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right]\mid^2 = & \frac{1}{n^2} \mid \sum_{i=1}^{n} \frac{\psi_s(Y_i)\,\delta_i \mathscr{W}_{n,i}(z)}{1-G(Y_i)} - \mathbb{E}\left[\sum_{i=1}^{n} \frac{\psi_s(Y_i)\,\delta_i \mathscr{W}_{n,i}(z)}{1-G(Y_i)}\right] \mid^2 \\
\leq & \frac{1}{n^2}\left(\sum_{i=1}^{n} \mathscr{W}_{n,i}(z)\right) \sum_{i=1}^{n} \mathscr{W}_{n,i}(z) \mid \frac{\psi_s(Y_i)\,\delta_i}{1-G(Y_i)} - \mathbb{E}\left[\frac{\psi_s(Y_i)\,\delta_i}{1-G(Y_i)}\right] \mid^2 \\
\leq & \frac{D}{n^2} \sum_{i=1}^{n} \mathscr{W}_{n,i}(z) \mid \frac{\psi_s(Y_i)\,\delta_i}{1-G(Y_i)} - \mathbb{E}\left[\frac{\psi_s(Y_i)\,\delta_i}{1-G(Y_i)}\right] \mid^2
\end{aligned}$$

By setting $A_i = \mid \frac{\psi_s(Y_i)\delta_i}{1-G(Y_i)} - \mathbb{E}\left[\frac{\psi_s(Y_i)\delta_i}{1-G(Y_i)}\right] \mid^2$, and under assumption $(F2)$ there exists a real number M such that $\sup_{s\in S} \mid \psi_s(x) \mid = \Psi(x) \leq M$, then under condition $(F5)$ $\mid \frac{\psi_s(Y_i)\delta_i}{1-G(Y_i)} \mid \leq \frac{M}{1-G(\tau)} \Rightarrow A_i \leq \left(\frac{2M}{1-G(\tau)}\right)^2 := C_0\, p.s.$. Hence

$$
\begin{aligned}
\sup_{s \in S} | \, \overline{r}_{\psi_s,n}(z) - \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right] |^2 &\leq \frac{D}{n^2} \max_i A_i \sum_{i=1}^{n} \mathscr{W}_{n,i}(z) \\
&\leq \frac{C_0 D}{n^2} \sum_{i=1}^{n} \mathscr{W}_{n,i}(z) \ p.s.
\end{aligned}
$$

We finally obtain

$$
\mathscr{P}_n(z) \leq \int_{\mathbb{R}^d} \frac{C_0 D}{n^2} \sum_{i=1}^{n} \mathscr{W}_{n,i}(z) \, \mu(dz) \ p.s. \tag{16}
$$

Under assumption (v) and by the dominated convergence theorem, the right-hand side of inequality 16 converges almost surely to zero. ■

Lemma 3. *If the assumptions* (v) *and* $(F1 - F5)$ *are satisfied. Then, for sufficiently large* n*,*

$$
\mathscr{Q}_n(z) = o(1) \tag{17}
$$

Proof We have

$$
\begin{aligned}
\mathscr{Q}_n(z) &= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \mathbb{E}\left[\overline{r}_{\psi_s,n}(z)\right] - r_{\psi_s}(z) \mid^2 \mu(dz) \\
&= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} \frac{\delta_i \psi_s(Y_i)}{1-G(Y_i)} \mathscr{W}_{n,i}(z)\right] - r_{\psi_s}(z) \mid^2 \mu(dz) \\
&\leq \int_{\mathbb{R}^d} \sup_{s\in S} \mid \mathbb{E}\left\{\sum_{i=1}^{n} \mathscr{W}_{n,i}(z)\, \mathbb{E}\big(\frac{\delta_1 \psi_s(Y_1)}{1-G(Y_1)} \mid Z_1 = z\big) - r_{\psi_s}(z) \mid^2 \mu(dz)\right\} \\
&= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \mathbb{E}\left\{\sum_{i=1}^{n} \mathscr{W}_{n,i}(z)\, \mathbb{E}\big(\frac{\psi_s(X_1)}{1-G(X_1)} \mathbb{I}_{\{X_1\leq C\}} \mid Z_1 = z\big) - r_{\psi_s}(z) \mid^2 \mu(dz)\right\} \\
&= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \mathbb{E}\left\{\sum_{i=1}^{n} \mathscr{W}_{n,i}(z) \frac{\psi_s(X_1)}{1-G(X_1)} \mathbb{E}\big(\mathbb{I}_{\{X_1\leq C\}} \mid X_1, Z_1 = z\big) - r_{\psi_s}(z) \mid^2 \mu(dz)\right\} \\
&= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \mathbb{E}\left\{\sum_{i=1}^{n} \mathscr{W}_{n,i}(z) \frac{\psi_s(X_1)}{1-G(X_1)} \mathbb{E}\big(\mathbb{I}_{\{X_1\leq C_1\}} \mid X_1, Z_1 = z\big)\right\} - r_{\psi_s}(z) \mid^2 \mu(dz) \\
&= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \mathbb{E}\left\{\sum_{i=1}^{n} \mathscr{W}_{n,i}(z)\, \psi_s(X) \mid Z = z\right\} - r_{\psi_s}(z) \mid^2 \mu(dz) \\
&= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \sum_{i=1}^{n} \mathscr{W}_{n,i}(z)\, \mathbb{E}\left\{\psi(X) \mid Z = z\right\} - r_{\psi_s}(z) \mid^2 \mu(dz) \\
&= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \sum_{i=1}^{n} \mathscr{W}_{n,i}(z)\, r_{\psi_s}(z) - r_{\psi_s}(z) \mid^2 \mu(dz) \\
&= \int_{\mathbb{R}^d} \sup_{s\in S} \mid \left(\sum_{i=1}^{n} \mathscr{W}_{n,i}(z) - 1\right) r_{\psi_s}(z) \mid^2 \mu(dz)
\end{aligned}
$$

By conditions (ii),(v), and by the dominated convergence theorem, we obtain result 17. ■

5.2. Proof of the corollary 1

Proof. The proof of Corollary 1 is straightforward. It suffices to apply Theorem 1 to $\widetilde{r}_{\psi_s,n}$ with $\psi_s(x) = \mathbb{I}_{\{x\leq s\}}$ and $s \in]-\infty; \tau_F]$. ■

6. Conclusion

We proposed a nonparametric partitioning estimator of the regression function in the case where the covariate space is subject to a finite or countably infinite measurable partition. Almost sure and uniform convergence over the class $\mathcal{F}$ was established under suitable conditions, thereby enabling the estimation of the conditional distribution function. Our perspectives are to extend these results to broader families of functions $\mathcal{F}$, under minimal assumptions, to allow for the estimation of the conditional variance and higher-order conditional moments.

Appendix

Measurable and bounded envelope of a family of functions

Let $\mathcal{F} = \{f_s : \mathcal{X} \to \mathbb{R}, s \in S\}$ be a family of measurable functions defined on a measurable space $(\mathcal{X}, \mathcal{A})$.

We say that a family of functions $\mathcal{F}$ admits a measurable and bounded envelope if there exists a measurable function $\Psi : \mathcal{X} \to \mathbb{R}_+$ such that

$$| f_s(x) | \leq \Psi(x), \forall s \in S, \forall x \in \mathcal{X}$$

and such that Ψ is bounded, ie

$$\sup_{x \in \mathcal{X}} \Psi(x) < \infty$$

Dominated Convergence Theorem

Theorem 2. *Let $(\mathcal{X}, \mathcal{A}, \mu)$ be a measure space. Suppose $\{f_n\}_{n\geq 1}$ is a sequence of measurable functions such that:*
1. $f_n(x) \to f(x)$ almost everywhere (pointwise convergence),
2. there exists an integrable function $g : \mathcal{X} \to \mathbb{R}_+$ such that $| f_n(x) | \leq g(x)$ for all n and almost every x.
Then:

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$$

and f is integrable.

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References

Beirlant, J. and Györfii, L. (1998). On the asymptotic L_2-error in partitioning regression estimation. *Journal of Statistical Planning and Inference*, 71:93– 107.

Bosq, D. and Lecoutre, J. P. (1987). Théorie de l'estimation fonctionnelle. *Economica Paris*

Carbonez, A., Györfi, L., and van der Meulen, E. C. (1995). Partitioning estimates of a regression function under random censoring. *Statistics and Decisions*, 13:21–37.

Collomb, Gérard. (1977). Estimation non paramétrique de la régression par la méthode du noyau : propriété de convergence asymptotiquememt normale indépendante. *Annales scientifiques de l'Université de Clermont.* Mathématiques 65 (15), PP. 24-46.

Dabrowska, D. M., (1987). Nonparametric regression with censored survival data. *Scand. J. Statist.*, Rev., 14, PP. 181-197.

Deheuvels, P. and Einmhal, J. H. J. (2000). Functional limit laws for the increments of Kaplan-Meier product-limit processes and applications. *Ann. Probab.* 3, PP. 1301-1335

Devroye, L. and Györfii, L. (1983). Distribution-free exponential bound on the L1 error of partitioning estimates of a regression function. *In : Proceedings of the Fourth Pannonian Symposium on Mathematical Statistics, Konecny, F., Mogyoródi, J., and Wertz, W., editors, Akadémiai Kiadó, Budapest, Hungary*, pP. 67–76,

Devroye, L., Györfi, L., and Lugosi, G. (1996). *Probabilistic Theory of Pattern Recognition.* Springer-Verlag, New York.

Györfi, L., Schäfer, D., and Walk, H. (2002). *Relative stability of global errors in nonparametric function estimations*. IEEE Transactions on Information Theory, Vol. 48 (8), pp. 2230 - 2242, DOI: 10.1109/TIT.2002.800491

Kaplan, E. K. and Meier, P. (1958) Nonparametric estimation from incomplete observations. *J. Amer. Statist. Assoc.*, 53, pp. 457-481.

Kohler, M.,Máthé, K., and Pintér, (2002). Prediction from randomly right censored data. *Journal of Multivariate Analysis*, 80, pp. 73–100

Kossi Essona Gneyou (2005),Vitesse de Convergence de Certains Estimateurs de Kaplan-Meier de la Régression, *Afrika Statistika*, Vol.1 (1), pp. 77-92

Prakasa Rao, B.L.S. (1983) *Nonparametric Functional Estimation.* Academic Press, Orlando.

Tukey, J.,W. (1947). Nonparametric estimation II. Statistically Equivalent Blocks and Tolerance Regions-The Continuous Case. *Annals of Mathematical Statistics*, 18(4), pp. 529–539.

Tukey, J. W. (1961a). Discussion, emphasizing the connection between analysis of variance and spectrum analysis. *Technometrics*, 3, pp. 191-219.