Afrika Statistika

Afrika Statistika
Vol. 19 (4), 2024, pages 4173 - 4186.
DOI: https://dx.doi.org/10.16929/as/2024.4173.341

ISSN 2316-090X

Homogenization and Large deviation with a fractional diffusion process

Ramatoulaye Dabo [(1)], **Raphaël Diatta** [(1,*)] **and Clément Manga** [(1)]

[(1)] Assane Seck University , Department of Mathematics, B.P. 523, Ziguinchor, Senegal.

Received on September 05, 2025; Accepted on October 19, 2025; Published online on November 17, 2025

Abstract. We establish a large deviation principle for the stochastic differential equation (SDE) driven by a fractional Brownian motion for any Hurst parameter $H \in (0,1)$, when both parameters δ (homogenization parameter) and ϵ (the large deviation parameter) tend to zero. Here, the main assumption is that the ratio (δ/ϵ) also tends to zero. To make the associated rate function explicit, our approach is purely probabilistic.

Key words: fractional Brownian motion; homogenization; large deviations; stochastic differential equation.
AMS 2010 Mathematics Subject Classification Objects : 60F10; 60H15; 35K00; 35B27.

[(*)] Corresponding author Raphaël Diatta: raphdiatta@gmail.com
Ramatoulaye Dabo: daboramatoulaye@gmail.com
Clément Manga: cmanga@univ-zig.sn

Résumé. Nous établissons un principe de grandes déviations pour l'EDS dirigée par un mouvement brownien fractionnaire pour tout paramètre de Hurst $H \in (0, 1)$ lorsque les deux paramètres δ (paramètre d'homogénéisation) et ϵ (paramètre des grandes déviations) tendent vers zéro. Ici, l'hypothèse majeure est que le rapport (δ/ϵ) tend vers zéro. Afin de rendre explicite la fonction de taux associée, notre approche est purement probabiliste.

Presentations of authors.

Raphaël Diatta, Ph.D., is an assistant Professor of Probabilities and Statistics at Department of Mathematics, Assane Seck University of Ziguinchor, SENEGAL.

Ramatoulaye Dabo, M.Sc., is preparing a Ph.D. degree at Department of Mathematics under the supervision Pr. Clément Manga, Assane Seck University of Ziguinchor, SENEGAL.

Clément Manga, Ph.D., is an associated Professor of Probabilities and Statistics at Department of Mathematics Assane Seck University of Ziguinchor, SENEGAL

1. Introduction

Consider the following stochastic differential equation (SDE):

$$X_t^{x,\epsilon,\delta_\epsilon} = x + \sqrt{\epsilon^H} \int_0^t \sigma\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right) dB_s^H + \left(\frac{\epsilon}{\delta_\epsilon}\right)^H \int_0^t b\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right) ds + \int_0^t c\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right) ds, \quad (1)$$

whose parameters ϵ and δ_ϵ depending on ϵ are strictly positive and tend towards zero. Here ϵ is the parameter of large deviation and δ_ϵ is the homogenization parameter of the process $X_t^{x,\epsilon,\delta_\epsilon}$ defined on a white noise probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})), \mathbb{P})$ where $\mathcal{S}'(\mathbb{R})$ is a tempered distribution space called dual of Schwartz space on which $\mathcal{B}(\mathcal{S}'(\mathbb{R}))$ is a Borel algebra and where:

- ⋆ $\{B_t^H; t > 0, H \in]0; 1]\}$ is a 1-dimensional fractional Brownian motion;
- ⋆ $b, c, \sigma : [0; T] \times \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ are measurable functions, white noise integral (see Bender *and al.*(2003), Diatta *and al.*(2022), Siska *and al.*(2004)), bounded and periodic 1 in each direction.

The study of the asymptotic behavior of such stochastic differential equations driven by standard Brownian motion, has been explored by several authors Diedhiou *and al.*(2008), Diedhiou *and al.*(2007), Pardoux *and al.*(1999). These researchers, whose approaches are primarily probabilistic, rely on two key aspects: the large deviation principle Dembo *and al.*(1998) and homogenization theory.

In Diedhiou *and al.*(2008),the authors establish the large deviation principle using the homogenized coefficients previously derived by Pardoux *and al.*(1999). Apart from the paper by Diatta *and al.*(2022) that dealt with the large deviation principle for a SDE driven by a fractional Brownian motion, these authors deal with the two theories either separately or in combination, but only in the case of a standard Brownian motion. To the best of our knowledge, we have not found in the literature any work that established, on the one hand, the homogenization of a SDE driven by a fractional Brownian motion, nor, on the other hand, a coupling between homogenization and the large deviation principle for a SDE driven by a fraction Brownian motion. For these reasons, we became interested in these problems.

Our objective in this paper is to study the asymptotic behavior of this SDE using the coupling of homogenization and large deviation principle when $\left(\frac{\delta_\epsilon}{\epsilon}\right)$ tends to 0.

Our paper is organize as follows. Section 2 presents the essential tools needed to establish our results. Section 3 presents the various results, which are divided into two subsections: the first subsection is devoted exclusively to study the homogenization, while the second focuses on the coupling of the tow theories within the same equation.

2. Preliminaries

In the next section, we will study the asymptotic behavior of the process $\{X_t^{x,\epsilon,\delta_\epsilon}; t > 0\}$ solution of the stochastic differential equation (1).Therefore, we need to provide here some mathematical tools that we will use in the following. Thus, denote expectation with respect to $\mathbb{P}$ by $\mathbb{E}$ and the gradient operator by $\nabla.$, $\langle ., .\rangle$ the standard Euclidean inner product on $L^2(\mathbb{R})$, and $\|.\|_{L^2(\mathbb{R})}$ the associated norm defined by $\|f\|_{L^2(\mathbb{R})} = \left(\int_0^t |f|^2 ds\right)^{\frac{1}{2}}$. It is well know that $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, with $L^2(\mathbb{R})$ is the Hilbert space.

The infinitesimal generator of the $\mathcal{S}'(\mathbb{R})$-process $X_t^{x,\epsilon,\delta_\epsilon}$ is

$$L^x_{H,\epsilon,\delta_\epsilon} := Ht^{2H-1} a\left(\frac{x}{\delta_\epsilon^H}\right)\frac{\partial^2}{\partial x^2} + b\left(\frac{x}{\delta_\epsilon^H}\right)\frac{\partial}{\partial x} + \left(\frac{\delta_\epsilon}{\epsilon}\right)^H c\left(\frac{x}{\delta_\epsilon^H}\right)\frac{\partial}{\partial x},$$

(see Bender *and al.*(2003)) and tends to

$$L^x_H := Ht^{2H-1} a\left(\frac{x}{\delta_\epsilon^H}\right)\frac{\partial^2}{\partial x^2} + b\left(\frac{x}{\delta_\epsilon^H}\right)\frac{\partial}{\partial x} \tag{2}$$

And we give the following assumptions:

(H_1) $a = \sigma\sigma*$ is the continuous function and there exist a constant $M > 0$ such that $M^{-1} < a\left(\frac{x}{\delta_\epsilon^H}\right) < M$ and b is the centering function.

(H_2) $\lim\limits_{\epsilon \to 0}\left(\frac{\delta_\epsilon}{\epsilon}\right)^H = 0$, for $H > 0$.

For clarity, we first recall the definition and theorem of large deviation principle.

Definition 1. Let $X_t^{x,\epsilon}$ be a $L^2(\mathbb{R})$-valued random variable and let $\mathbb{P}_\epsilon$ denote its distribution on the Borel subsets of $L^2(\mathbb{R})$. The family $\{X^{x,\epsilon}; \epsilon \geqslant 0\}$ satisfies a *large deviation principle* (LDP) if there exists a lower semi-continuous function $I : L^2(\mathbb{R}) \longrightarrow [0, +\infty]$ such that

- for each open set $O \subseteq L^2(\mathbb{R})$
$$\liminf_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(X^{x,\epsilon} \in O\right) \geqslant -\inf_{x \in O} I(x),$$
- for each closed set $\Gamma \subseteq L^2(\mathbb{R})$
$$\limsup_{\epsilon \to 0} \epsilon \log \mathbb{P}\left(X^{x,\epsilon} \in \Gamma\right) \leqslant -\inf_{x \in \Gamma} I(x).$$

I is called the rate function for the LDP. A rate function is good if for each $\alpha \in \mathbb{R}_+,\ \{x : I(x) \leqslant \alpha\}$ is a compact subset of $L^2(\mathbb{R})$.

Let us define, for each $T > 0$ and $x \in L^2(\mathbb{R})$

$$g^\epsilon_{T,x}(\theta) = \epsilon \log \mathbb{E}\left[\exp\left(\frac{1}{\epsilon}\langle \theta, X_T^{x,\epsilon}\rangle\right)\right] \quad \epsilon > 0\ ,\ \theta \in L^2(\mathbb{R}).$$

$$g^{T,x}(\theta) = \lim_{\epsilon \to 0} g^\epsilon_{T,x}(\theta) = \lim_{\epsilon \to 0} \epsilon \log \mathbb{E}\left[\exp\left(\frac{1}{\epsilon}\langle \theta, X_T^{x,\epsilon}\rangle\right)\right]$$

when the limit exists with respect to $x \in L^2(\mathbb{R})$ (see Dembo *and al.*(1998), Freidlin *and al.*(1999)).

Theorem 1. *(see Dembo and al.(1998)) Fix $T > 0$ and $x \in L^2(\mathbb{R})$. Assume that*

1. *For each $\theta \in L^2(\mathbb{R}),\ g_{T,x}(\theta)$ is well defined in $[-\infty; +\infty]$.*
2. *The origin is in the interior of the set $\left\{\theta \in L^2(\mathbb{R}) : g_{T,x}(\theta) < +\infty\right\}$.*
3. *The set $O = \left\{\theta \in L^2(\mathbb{R}) : |g_{T,x}(\theta)| < +\infty\right\}$ has a non-empty interior $\overset{\circ}{O}$, $\nabla g_{T,x}(\theta)$ is well defined for all $\theta \in \overset{\circ}{O}$, and*
$$\lim_{\theta \to \partial O, \theta \in \overset{\circ}{O}} \sup \|\nabla g_{T,x}(\theta)\| = \infty.$$

Then the random variables $\{X_T^{x,\epsilon} : \epsilon > 0\}$ satisfy an LDP with good rate function $I_{T,x}$ defined by
$$I_{T,x}(z) = \sup_{\theta \in L^2(\mathbb{R})} \left\{\langle \theta, z\rangle - g_{T,x}(\theta)\right\},\ z \in L^2(\mathbb{R}).$$

3. Main results

In this section, we will deal with homogenization in the first subsection, so that the SDE will depend only on δ. In the second subsection, we will carry out the coupling.

3.1. Homogenization

In this subsection, we use the same approaches as in Diedhiou *and al.*(2007), Pardoux *and al.*(1999), Stroock *and al.*(1988) to analyze the limiting behavior of the process $\{X_t^{x,\epsilon,\delta_\epsilon}, t>0\}$ solution of the SDE (1), This analysis is carried out under a centering condition imposed on the drift term b, within the framework of a homogenization procedure as δ_ϵ tens to 0 assuming that $\epsilon=1$. In this case, the process $\{X_t^{x,\epsilon,\delta_\epsilon}; t>0\}$ reduces to $\{X_t^{x,\delta}, t>0\}$ who takes the following form:

$$X_t^{x,\delta} = x + \int_0^t \sigma\left(\frac{X_s^{x,\delta}}{\delta^H}\right) dB_s^H + \frac{1}{\delta^H}\int_0^t b\left(\frac{X_s^{x,\delta}}{\delta^H}\right) ds + \int_0^t c\left(\frac{X_s^{x,\delta}}{\delta^H}\right) ds. \tag{3}$$

Theorem 2. *We assume that for all $H\in(0,1)$ and $t>0$, $\widetilde{X}_t^{\delta^H} = \frac{1}{\delta^H}X_t^{x,\delta}$. Then there exists a fractional Brownian motion $\widetilde{B}_t^{\delta^H} := \left(\frac{1}{\delta^H}\right) B_{\delta t}^H$ such that*

$$\widetilde{X}_t^{\delta^H} = \frac{x}{\delta^H} + \int_0^t \sigma\left(\widetilde{X}_s^{\delta^H}\right) d\widetilde{B}_s^{\delta^H} + 2H\int_0^t s^{2H-1} b\left(\widetilde{X}_s^{\delta^H}\right) ds + 2H\delta^H \int_0^t s^{2H-1} c\left(\widetilde{X}_s^{\delta^H}\right) ds \tag{4}$$

whose associated generator is $\widetilde{L}_{\delta^H} = 2Ht^{2H-1}\widetilde{L}_x$ where

$$\widetilde{L}_x = \frac{1}{2}a\left(\frac{x}{\delta^H}\right)\frac{\partial^2}{\partial x^2} + b\left(\frac{x}{\delta^H}\right)\frac{\partial}{\partial x} + \delta^H c\left(\frac{x}{\delta^H}\right)\frac{\partial}{\partial x}.$$

proof. Let $\widetilde{X}_t = \frac{1}{\delta^H}X_t^{x,\delta}$ and show that $\widetilde{X}_t = \widetilde{X}_t^{\delta^H}$ (4). We then write

$$\begin{aligned}\widetilde{X}_t = \frac{1}{\delta^H}X_t^{x,\delta} &= \frac{x}{\delta^H} + \int_0^t \sigma\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\tfrac{1}{\delta^H}B_s^H] + \int_0^t b\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\tfrac{s}{\delta^{2H}}] \\ &+ \delta^H \int_0^t c\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\tfrac{s}{\delta^{2H}}]\end{aligned} \tag{5}$$

a. We begin by showing that $\widetilde{X}_t \le \widetilde{X}_t^{\delta^H}$

In this section, we distinguish two cases according to H:
Case a_1: $H\in(\frac{1}{2};1)$
Here, we also consider two separate cases depending on s

- $0 < s < 1$, replacing B_s^H with $B^H_{s^{\frac{1}{2H}}}$ in (5) since the latter is larger in law and by setting $\widetilde{X}_t = \frac{1}{\delta^H} X^{x,\delta}_{(\delta t)^{2H}}$, we have

$$\widetilde{X}_t \leq \frac{x}{\delta^H} + \int_0^{(\delta t)^{2H}} \sigma\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\frac{1}{\delta^H} B^H_{s^{\frac{1}{2H}}}] + \int_0^{(\delta t)^{2H}} b\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\left(\frac{s^{\frac{1}{2H}}}{\delta}\right)^{2H}]$$
$$+ \delta^H \int_0^{(\delta t)^{2H}} c\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\left(\frac{s^{\frac{1}{2H}}}{\delta}\right)^{2H}]$$

By setting $r = \frac{1}{\delta} s^{\frac{1}{2H}}$ we obtain

$$\widetilde{X}_t \leq \frac{x}{\delta^H} + \int_0^t \sigma\left(\frac{X^{x,\delta}_{(\delta r)^{2H}}}{\delta^H}\right) d[\frac{1}{\delta^H} B^H_{\delta r}] + \int_0^t b\left(\frac{X^{x,\delta}_{(\delta r)^{2H}}}{\delta^H}\right) d(r^{2H})$$
$$+ \delta^H \int_0^t c\left(\frac{X^{x,\delta}_{(\delta r)^{2H}}}{\delta^H}\right) d(r^{2H})$$
$$\widetilde{X}_t \leq \frac{x}{\delta^H} + \int_0^t \sigma\left(\widetilde{X}_r^{\delta^H}\right) d\widetilde{B}_r^{\delta^H} + 2H \int_0^t r^{2H-1} b\left(\widetilde{X}_r^{\delta^H}\right) dr + 2H\delta^H \int_0^t r^{2H-1} c\left(\widetilde{X}_r^{\delta^H}\right) dr$$
$$= \widetilde{X}_t^{\delta^H}$$

so we have $\frac{1}{\delta^H} B^H_{\delta r}$, which we denote $\widetilde{B}_r^{\delta^H}$, and we deduce $\widetilde{X}_t \leq \widetilde{X}_t^{\delta^H}$.

- $1 < s \Rightarrow s < s^{2H}$, replacing s with s^{2H} and by setting $\widetilde{X}_t = \frac{1}{\delta^H} X^{x,\delta}_{\delta t}$ in (5), we obtain:

$$\widetilde{X}_t \leq \frac{x}{\delta^H} + \int_0^{\delta t} \sigma\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\frac{1}{\delta^H} B_s^H] + \int_0^{\delta t} b\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\left(\frac{s}{\delta}\right)^{2H}] + \delta^H \int_0^{\delta t} c\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\left(\frac{s}{\delta}\right)^{2H}]$$

By setting $r = \frac{s}{\delta}$, we obtain the same $\widetilde{B}_r^{\delta^H}$, and we deduce $\widetilde{X}_t \leq \widetilde{X}_t^{\delta^H}$.

Case a_2: $H \in (0; \frac{1}{2})$

In this case, the same demonstration applies by interchanging the considerations related to time s.

Hence, for every $H \in (0; 1)$, $\widetilde{X}_t \leq \widetilde{X}_t^{\delta^H}$.

b. We now establish that for all $H \in (0; 1)$, $\widetilde{X}_t \geq \widetilde{X}_t^{\delta^H}$.

Similarly, we consider separate cases depending on the values of H.

case b_1: $H \in (0; \frac{1}{2})$

The time s is treated separately in each case.

- $1 < s \Rightarrow s^{2H} < s$, replacing s with s^{2H} and put $\widetilde{X}_t = \frac{1}{\delta^H} X^{x,\delta}_{\delta t}$ (5), we have

$$\widetilde{X}_t \geq \frac{x}{\delta^H} + \int_0^{\delta t} \sigma\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\frac{1}{\delta^H} B_s^H] + \int_0^{\delta t} b\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\left(\frac{s}{\delta}\right)^{2H}]$$
$$+ \delta^H \int_0^{\delta t} c\left(\frac{X_s^{x,\delta}}{\delta^H}\right) d[\left(\frac{s}{\delta}\right)^{2H}]$$

By setting $r = \frac{s}{\delta}$, we obtain the same $\widetilde{B}_r^{\delta^H} = \frac{1}{\delta^H} B_{\delta r}^H$, and

$$\widetilde{X}_t \geq \frac{x}{\delta^H} + \int_0^t \sigma\left(\widetilde{X}_r^{\delta^H}\right) d\widetilde{B}_r^{\delta^H} + 2H \int_0^t r^{2H-1} b\left(\widetilde{X}_r^{\delta^H}\right) dr + 2H\delta^H \int_0^t r^{2H-1} c\left(\widetilde{X}_r^{\delta^H}\right) dr$$

We deduce $\widetilde{X}_t \geq \widetilde{X}_t^{\delta^H}$.

- $0 < s < 1$, replacing B_s^H with $B^H_{s^{\frac{1}{2H}}}$ in (5) since the process $B^H_{s^{\frac{1}{2H}}}$ is smaller in distribution than B_s^H and by setting $\widetilde{X}_t = \frac{1}{\delta^H} X^{x,\delta}_{(\delta t)^{2H}}$ and $r = \frac{1}{\delta} s^{\frac{1}{2H}}$,we deduce $\widetilde{X}_t \geq \widetilde{X}_t^{\delta^H}$.

case b_2: $H \in (\frac{1}{2}; 1)$

By making the step changes in time s from the previous case, the same reasoning applies.

Hence $\widetilde{X}_t \geq \widetilde{X}_t^{\delta^H}$ for every $H \in (0; 1)$.

Finally, $\widetilde{X}_t = \widetilde{X}_t^{\delta^H}$, and by applying the fractional Itô formula to a function $F \in C^{1,2}$ associated with $\widetilde{X}_t^{\delta^H}$, we obtain the operator $\widetilde{L}_{\delta^H}$. ■

We consider the process $\widetilde{X}_t^{\delta^H}$ in 1-dimensional torus $L^2(\mathbb{T})$ and for $\widetilde{L}_{\delta^H} \in L^2(\mathbb{T})$ we note L_H the operator defined by

$$L_H := Ht^{2H-1} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}. \tag{6}$$

Furthermore, for $f \in L^2(\mathbb{T})$, note

$$[P_t f](x) = \mathbb{E}[f(\widetilde{X}_t)] \tag{7}$$

We have the following propositions and lemmas:

Proposition 1. *For all $\delta > 0$, the $L^2(\mathbb{T})$ -valued diffusion process $\widetilde{X}_t^{\delta^H}$ of generator $\widetilde{L}_{\delta^H}$ has a unique invariant probability μ_δ*

Lemma 1. *Under the assumptions (H_1), $x \in L^2(\mathbb{T})$ and for all $t \geq 0$ and for $f \in C^2(\mathbb{T})$, we have*

$$\partial_t [P_t f](x) = L_H [P_t f](x) \tag{8}$$

Proof As $\delta \longrightarrow 0$, clearly $\widetilde{X}_t^{\delta^H} \longrightarrow \widetilde{X}_t \in L^2(\mathbb{T})$. By using Ito's formula to $f(\widetilde{X}_t^{\delta^H})$ and the derivation with respect to t we obtain, when δ tends to 0,

$$\partial_t [P_t f](x) = \mathbb{E}[\widetilde{L}_H f(\widetilde{X}_t)] = L_H [P_t f](x). \ ■$$

Proposition 2. *For any $\delta > 0$ and $f \in L^2(\mathbb{T})$, there exists a positive constant ρ such that*

$$|\mathbb{E}[f(\widetilde{X}_t^{\delta^H})] - \int f(x)\mu_\delta(dx)| \leq \|f\|_{L^2(\mathbb{T})}e^{-\rho t^{2H}} \tag{9}$$

If f is centered with respect to $\mu_\delta$then we get

$$|\mathbb{E}[f(\widetilde{X}_t^{\delta^H})]| \leq \|f\|_{L^2(\mathbb{T})}e^{-\rho t^{2H}} \tag{10}$$

Lemma 2. *There exists a constant K such that for any periodic bounded bored function f on $L^2(\mathbb{T})$, we have*

$$\|f - m(f)\|_{L^2(\mathbb{T})} \leq K\|f'\|_{L^2(\mathbb{T})} \tag{11}$$

where $m(f) = \int_0^1 f(x)dx$

Proof. Let t and $s \in [0;1]$, we have $f(\widetilde{X}_t^{\delta^H}) - f(\widetilde{X}_s^{\delta^H}) = \int_s^t f'(\widetilde{X}_r^{\delta^H})dr$.
We integrate for $s \in [0;1]$, $m(f) = \int_0^1 f(\widetilde{X}_s^{\delta^H})ds = f(\widetilde{X}_t^{\delta^H}) - \int_0^1 \int_s^t f'(\widetilde{X}_r^{\delta^H})drds$.

By using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|f(\widetilde{X}_t^{\delta^H}) - m(f)|^2 \leq \int_0^1 |t-s| \int_s^t |f'(\widetilde{X}_r^{\delta^H})|^2 dr \leq \int_0^1 |f'(\widetilde{X}_r^{\delta^H})|^2 dr \\
\int_0^1 |f(\widetilde{X}_t^{\delta^H}) - m(f)|^2 dt \leq \int_0^1 |f'(\widetilde{X}_r^{\delta^H})|^2 dr \\
\|f - m(f)\|_{L^2(\mathbb{T})} \leq \|f'\|_{L^2(\mathbb{T})}
\end{aligned} \tag{12}$$

So, for all t and $s \in [0;T]$, there exists K, $\|f - m(f)\|_{L^2(\mathbb{T})} \leq K\|f'\|_{L^2(\mathbb{T})}$. ■

We now proceed to prove the preceding proposition.

Proof. Put $[P_tf](x) = \mathbb{E}[f(\widetilde{X}_t)]$ and $g(t) = \int_0^1([P_tf](x))^2dx$

$$\begin{aligned}
\partial_t g(t) &= 2\int_0^1 [P_tf](x)\partial_t[P_tf](x)dx = 2\int_0^1 [P_tf](x)L_H[P_tf](x)dx \\
&= 2Ht^{2H-1}\int_0^1 [P_tf](x)\partial_x(a(x)\partial_x[P_tf](x))dx \\
&= -2Ht^{2H-1}\int_0^1 \partial_x[P_tf]^2(x)a(x)dx
\end{aligned} \tag{13}$$

By assumption (H_1), previous lemma, and Gronwall's lemma, we have

$$\partial_t g(t) \leq -2Ht^{2H-1}(MK)^{-1}\int_0^1 |[P_t f](x)|^2 dx = -2Ht^{2H-1}\rho g(t)$$
$$g(t) \leq g(0)\exp(-\rho t^{2H}) = \int_0^1 |f(x)|^2 dx \exp(-\rho t^{2H})$$
$$\int_0^1 ([P_t f](x))^2 dx \leq \int_0^1 |f(x)|^2 dx \exp(-\rho t^{2H})$$
$$\iff \|[P_t f](x)\|_{L^2(\mathbb{T})} \leq \|f\|_{L^2(\mathbb{T})}\exp(-\rho t^{2H})$$

We deduce that $|\mathbb{E}[f(\widetilde{X}_t^{\delta^H})] - \int f(x)\mu_\delta(dx)| \leq \|f\|_{L^2(\mathbb{T})} e^{-\rho t^{2H}}$. ■

3.2. Large deviation with a homogenized fractional diffusion process

In order to state our main theorem, we need to define the solution of Poisson equation. For this purpose, we refer to the relevant articles Diedhiou *and al.*(2007), Pardoux *and al.*(1999). Let $\widehat{b} = \int_0^{+\infty} \mathbb{E}_x[b(\frac{X_t^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H})]dt$ denote the solution of Poisson equation $L_H^x \widehat{b} + b = 0$, which satisfies the centering condition $\int_{L^2(\mathbb{T})} \widehat{b}(z)\mu(dz) = 0$ for $\widehat{b} \in C^\infty(\mathbb{T})$ and is the periodic function of x. We set $\widetilde{X}_t^{\delta_\epsilon^H} = \frac{1}{\delta_\epsilon^H} X_t^{x,\epsilon,\delta_\epsilon}$, by using Ito's fractional formula to the process $\widehat{b}(\widetilde{X}_t^{\delta_\epsilon^H})$ and adding $X_t^{x,\epsilon,\delta_\epsilon}$ solution of (1) we have

$$X_t^{x,\epsilon,\delta_\epsilon} + \delta_\epsilon^H \left(\widehat{b}(\widetilde{X}_t^{\delta_\epsilon^H}) - \widehat{b}(\frac{x}{\delta_\epsilon^H})\right) = x + \sqrt{\epsilon^H}\int_0^t (I+\nabla\widehat{b})\sigma(\widetilde{X}_s^{\delta_\epsilon^H})dB_s^H + \int_0^t (I+\nabla\widehat{b})C(\widetilde{X}_s^{\delta_\epsilon^H})ds \tag{14}$$

and subsequently, we obtain

$$\frac{X_t^{x,\epsilon,\delta_\epsilon}}{\epsilon^H} + \left(\frac{\delta_\epsilon}{\epsilon}\right)^H \left(\widehat{b}(\widetilde{X}_t^{\delta_\epsilon^H}) - \widehat{b}(\frac{x}{\delta_\epsilon^H})\right) = \frac{x}{\epsilon^H} + \left(\frac{\delta_\epsilon}{\epsilon}\right)^H \int_0^t (I+\nabla\widehat{b})\sigma(\widetilde{X}_s^{\delta_\epsilon^H})d[\left(\frac{\sqrt{\epsilon}}{\delta_\epsilon}\right)^H B_s^H] + \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}\int_0^t (I+\nabla\widehat{b})C(\widetilde{X}_s^{\delta_\epsilon^H})d[\left(\frac{\sqrt{\epsilon}}{\delta_\epsilon}\right)^{2H} s]. \tag{15}$$

By applying the same reasoning as in the previous theorem 2, we have

$$\frac{X_t^{x,\epsilon,\delta_\epsilon}}{\epsilon^H} = \frac{x}{\epsilon^H} + \left(\frac{\delta_\epsilon}{\epsilon}\right)^H \int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t} (I+\nabla\widehat{b})\sigma(\widetilde{X}_t^{\delta_\epsilon^H})d\widetilde{B}_s^{\delta_\epsilon^H} + 2H\left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}\int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t} s^{2H-1}(I+\nabla\widehat{b})C(\widetilde{X}_s^{\delta_\epsilon^H})ds - \left(\frac{\delta_\epsilon}{\epsilon}\right)^H\left(\widehat{b}(\widetilde{X}_t^{\delta_\epsilon^H}) - \widehat{b}(\frac{x}{\delta_\epsilon^H})\right). \tag{16}$$

We define

$$\bar{A} = \int_{L^2(\mathbb{T})} (I + \nabla\widehat{b})a(I + \nabla\widehat{b})^* \mu(dz) \qquad \textit{and} \qquad \bar{C} = \int_{L^2(\mathbb{T})} (I + \nabla\widehat{b})C(z)\mu(dz)$$

and consequently, we have the following notions:

Proposition 3. *The following convergence hold in probability when δ_ϵ tends to 0:*

i) $2H \int_0^t s^{2H-1}(I + \nabla\widehat{b})a_s(I + \nabla\widehat{b})^*(\widetilde{X}_s^{\delta_\epsilon^H})ds \longrightarrow t^{2H}\bar{A}$
ii) $2H \int_0^t s^{2H-1}(I + \nabla\widehat{b})C(\widetilde{X}_s^{\delta_\epsilon^H})ds \longrightarrow t^{2H}\bar{C}$

Proof: We have

i) Let's see the difference

$$\begin{aligned}
&2H \int_0^t s^{2H-1}(I + \nabla\widehat{b})a_s(I + \nabla\widehat{b})^*(\widetilde{X}_s^{\delta_\epsilon^H})ds - t^{2H}\bar{A} \\
&= \int_0^t (I + \nabla\widehat{b})a_s(I + \nabla\widehat{b})^*(\widetilde{X}_s^{\delta_\epsilon^H})d(s^{2H}) - t^{2H}\int_{L^2(\mathbb{T})} (I + \nabla\widehat{b})a(I + \nabla\widehat{b})^*(x)\mu(dz)
\end{aligned}$$

Set $f(z) = (I + \nabla\widehat{b})a(I + \nabla\widehat{b})^*(z)$ and $\bar{f}\left(\frac{X_t^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right) = f\left(\frac{X_t^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right) - \int_{L^2(\mathbb{T})} f(x)\mu_{\delta_\epsilon}(dx)$, we have
$\int_0^t \bar{f}\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right)d(s^{2H}) = \int_0^t f\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right)d(s^{2H}) - t^{2H}\int_{L^2(\mathbb{T})} f(x)\mu_{\delta_\epsilon}(dx)$. Since $\mu_{\delta_\epsilon} \longrightarrow \mu$ in $L^2(\mathbb{T})$, it suffices to show that $\int_0^t \bar{f}\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right)d(s^{2H}) \longrightarrow 0$ in probability as $\left(\frac{\delta_\epsilon}{\epsilon}\right)^H \longrightarrow 0$.

Now $\widetilde{X}_t^{\delta_\epsilon^H} = \frac{1}{\delta_\epsilon^H}X^{x,\epsilon,\delta_\epsilon}_{\frac{\delta_\epsilon}{\sqrt{\epsilon}}t}$ and put $s = \frac{\sqrt{\epsilon}}{\delta_\epsilon}r$, consequently
$\int_0^t \bar{f}\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right)d(s^{2H}) = \left(\frac{\delta_\epsilon}{\sqrt{\epsilon}}\right)^{2H}\int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t}\bar{f}(\widetilde{X}_r^{\delta_\epsilon^H})d(r^{2H})$
By Cauchy-Schwartz's inequality, we obtain

$$\mathbb{E}[\left(\frac{\delta_\epsilon}{\sqrt{\epsilon}}\right)^{2H}\int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t}\bar{f}(\widetilde{X}_r^{\delta_\epsilon^H})d(r^{2H})] \leq \left(\sqrt{\epsilon}\right)^H\left(\frac{\delta_\epsilon}{\epsilon}\right)^H t^H \times \|f\|_2 \longrightarrow 0$$

as $\left(\frac{\delta_\epsilon}{\epsilon}\right)^H \longrightarrow 0$. Hence $\int_0^t f\left(\frac{X_s^{x,\epsilon,\delta_\epsilon}}{\delta_\epsilon^H}\right)ds \longrightarrow t^{2H}\int_{L^2(\mathbb{T})} f(x)\mu(dx)$

ii) Identical to i), but the result was proven by Pardoux Diedhiou *and al.*(2007), Pardoux *and al.*(1999). ■

Now let us define for each $T > 0$ and $x \in L^2(\mathbb{T})$,

$$\begin{aligned} g_{T,x}(\theta) := \lim_{\epsilon\to 0} g^{\epsilon}_{T,x}(\theta) = \lim_{\epsilon\to 0} \epsilon^H \log \mathbb{E}\left[\exp\left\{\frac{1}{\epsilon^H}\langle X^{x,\epsilon,\delta_\epsilon}_T, \theta\rangle\right\}\right] \\ = \langle \theta, x\rangle + T^{2H}\left(\frac{1}{2}\langle \bar{A}\theta, \theta\rangle + \langle \bar{C}\theta, \theta\rangle\right) \end{aligned} \tag{17}$$

Under (H_1), $g_{T,x}$ is well defined in $[-\infty, +\infty]$ and $\left\{\theta \in L^2(\mathbb{R}) : |g_{T,x}(\theta)| < +\infty\right\}$ is a compact subset of $L^2(\mathbb{T})$. Our main theorem is as follows:

Theorem 3. *For all $H \in (0, 1)$ and $T > 0$, we assume that the hypothesis (H_1) and (H_2) hold true, then for $x \in L^2(\mathbb{R})$ the family $\left\{X^{x,\epsilon,\delta_\epsilon}_T : \epsilon > 0\right\}$ of random variables satisfies a LDP with good rate function $I_{T,x}$ defined by*

$$I_{T,x}(z) = \sup_{\{\theta\in\mathbb{R}\}} \left\{\langle\theta, z\rangle - g_{T,x}(\theta)\right\},\ z \in L^2(\mathbb{R}). \tag{18}$$

Proof. According to (16),

$$\begin{aligned} g^{\epsilon}_{T,x}(\theta) =& \epsilon^H \log \mathbb{E}\left[\exp\left\{\langle\theta, \frac{X^{x,\epsilon,\delta_\epsilon}_t}{\epsilon^H}\rangle\right\}\right] \\ =& \epsilon^H \log \bar{\mathbb{E}}[\exp\{\frac{1}{\epsilon^H}\langle x, \theta\rangle + \left(\frac{\delta_\epsilon}{\epsilon}\right)^H \int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t} \langle (I + \nabla\widehat{b}(\widetilde{X}^{\delta_\epsilon^H}_s))\sigma(\widetilde{X}^{\delta_\epsilon^H}_s) d\widetilde{B}^{\delta_\epsilon^H}_s, \theta\rangle \\ &+ \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H} 2H \int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t} s^{2H-1}\langle (I + \nabla\widehat{b}(\widetilde{X}^{\delta_\epsilon^H}_s))c(\widetilde{X}^{\epsilon^H}_s) ds, \theta\rangle \\ &- \left(\frac{\delta_\epsilon}{\epsilon}\right)^H \langle\left(\widehat{b}(\widetilde{X}^{\delta_\epsilon^H}_t) - \widehat{b}(\frac{x}{\delta_\epsilon^H})\right), \theta\rangle\}],\ \theta > 0 \end{aligned} \tag{19}$$

Using the Girsanov fractional formula Bender *and al.*(2003), we obtain

$$\begin{aligned} g^{\epsilon}_{T,x}(\theta) = \langle\theta, x\rangle + \epsilon^H \log \bar{\mathbb{E}}[\exp\{\left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H} H \int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t} s^{2H-1}\langle (I + \nabla\widehat{b}(\widetilde{X}^{\delta_\epsilon^H}_s))\sigma(\widetilde{X}^{\delta_\epsilon^H}_s), \theta\rangle^2 ds \\ + \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H} 2H \int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t} s^{2H-1}\langle (I + \nabla\widehat{b}(\widetilde{X}^{\delta_\epsilon^H}_s))c(\widetilde{X}^{\epsilon^H}_s), \theta\rangle ds \\ - \left(\frac{\delta_\epsilon}{\epsilon}\right)^H \langle\left(\widehat{b}(\widetilde{X}^{\delta_\epsilon^H}_t) - \widehat{b}(\frac{x}{\delta_\epsilon^H})\right), \theta\rangle\}],\ \theta > 0 \end{aligned} \tag{20}$$

where $\bar{\mathbb{E}}$ is the expectation with respect to the probability $\bar{\mathbb{P}}$ such that

$$\begin{aligned}\frac{d\bar{\mathbb{P}}}{d\mathbb{P}} =& \exp\{\left(\frac{\delta_\epsilon}{\epsilon}\right)^H \int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t} \langle (I+\nabla\widehat{b})\sigma, \theta\rangle d\widetilde{B}_s^{\delta_\epsilon^H} \\ & - \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H} H \int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t} s^{2H-1}\langle (I+\nabla\widehat{b})\sigma, \theta\rangle^2 ds\}\end{aligned} \tag{21}$$

Let us set

$$\Phi(z) = 2HT^{2H-1}\left[\frac{1}{2}\langle (I+\nabla\widehat{b})\sigma(z), \theta\rangle^2 + \langle (I+\nabla\widehat{b})C(z), \theta\rangle\right] = 2HT^{2H-1}\varphi(z)$$

where $\varphi(z) = \frac{1}{2}\langle (I+\nabla\widehat{b})\sigma(z), \theta\rangle^2 + \langle (I+\nabla\widehat{b})C(z), \theta\rangle$
Let $\Psi^{\delta_\epsilon} \in C^\infty(\mathbb{T})$ be the solution of $\widetilde{L}_{\delta_\epsilon^H}\Psi^{\delta_\epsilon} = 2HT^{2H-1}\widetilde{L}_x\Theta^{\delta_\epsilon}$

$\widetilde{L}_x\Theta^{\delta_\epsilon} = \varphi^{\delta_\epsilon} - \int_{L^2(\mathbb{T})}\varphi^{\delta_\epsilon}(z)\mu_{\delta_\epsilon}(dz)$, which satisfies $\int_0^t \Theta^{\delta_\epsilon}(z)\mu_{\delta_\epsilon}(dz) = 0$.
Applying Ito formula to $\left(\frac{\delta_\epsilon}{\sqrt{\epsilon}}\right)^{2H}\Psi^{\delta_\epsilon}(\widetilde{X}^{\delta_\epsilon^H}_{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t})$ then putting this into the expression (20) we have

$$\begin{aligned}g^\epsilon_{T,x}(\theta) =& \langle\theta, x + T^{2H}\int_{L^2(\mathbb{T})}\varphi^{\delta_\epsilon}(z)\mu_\epsilon(dz)\rangle + \epsilon^H \log\bar{\mathbb{E}}[\exp\{\langle\left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}\left(\Psi^{\delta_\epsilon}(\widetilde{X}^{\delta_\epsilon^H}) - \Psi^{\delta_\epsilon}(\widetilde{X}_0^{\delta_\epsilon^H})\right) \\ & - \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}\int_0^{\left(\frac{\sqrt{\epsilon}}{\delta_\epsilon}\right)t}\sigma\nabla\Psi^\epsilon(\widetilde{X}^{\delta_\epsilon^H}))d\widetilde{B}_s^{\delta_\epsilon^H} + \left(\frac{\delta_\epsilon}{\epsilon}\right)^H\left(\widehat{b}(\widetilde{X}^{\delta_\epsilon^H}) - \widehat{b}(\frac{x}{\delta_\epsilon^H})\right), \theta\rangle\}]\end{aligned} \tag{22}$$

Finally, we estimate this last term written with respect to the expectation

$$\begin{aligned}&\epsilon^H \log\bar{\mathbb{E}}[\exp\{-\left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}\int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t}\sigma(\widetilde{X}^{\delta_\epsilon^H})\nabla\Psi^{\delta_\epsilon}(\widetilde{X}^{\delta_\epsilon^H}))d\widetilde{B}_s^{\delta_\epsilon^H} \\ &+ \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}\left(\Psi^{\delta_\epsilon}(\widetilde{X}^{\delta_\epsilon^H}) - \Psi^{\delta_\epsilon}(\widetilde{X}_0^{\delta_\epsilon^H})\right) + \left(\frac{\delta_\epsilon}{\epsilon}\right)^H\left(\widehat{b}(\widetilde{X}^{\delta_\epsilon^H}) - \widehat{b}(\frac{x}{\delta_\epsilon^H})\right)\}] \\ &\le \epsilon^H \log\bar{\mathbb{E}}[\exp\{H\left(\frac{\delta_\epsilon}{\epsilon}\right)^{4H}\int_0^{\frac{\sqrt{\epsilon}}{\delta_\epsilon}t}s^{2H-1}\sigma\nabla\Psi^{\delta_\epsilon}(\widetilde{X}^{\delta_\epsilon^H})ds + \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}K_2 + \left(\frac{\delta_\epsilon}{\epsilon}\right)^H K_3\}] \\ &\le \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}\frac{T^{2H}}{2}\|\sigma\nabla\Psi^\epsilon\|^2 + \epsilon^H\left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}K_2 + \epsilon^H\left(\frac{\delta_\epsilon}{\epsilon}\right)^H K_3 \\ &\le \left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}K_1 + \epsilon^H\left(\frac{\delta_\epsilon}{\epsilon}\right)^{2H}K_2 + \epsilon^H\left(\frac{\delta_\epsilon}{\epsilon}\right)^H K_3\end{aligned}$$

This term tends to zero when $\left(\frac{\delta_\epsilon}{\epsilon}\right)^H$ tends to zero. So we have from (22)

$$\begin{aligned} g_{T,x}(\theta) &= \lim_{\epsilon\to 0} g^{\epsilon}_{T,x}(\theta) = \langle\theta, x\rangle + T^{2H}\int_{L^2(\mathbb{T})}\varphi(z)\mu(dz) \\ &= \langle\theta, x\rangle + T^{2H}\int_{L^2(\mathbb{T})}\left[\frac{1}{2}\langle (I+\nabla\widehat{b})\sigma(z),\theta\rangle^2 + \langle (I+\nabla\widehat{b})C(z),\theta\rangle\right]\mu(dz) \end{aligned}$$

$$g_{T,x}(\theta) = \langle\theta, x\rangle + T^{2H}\left(\frac{1}{2}\langle\bar{A}\theta,\theta\rangle + \langle\bar{C},\theta\rangle\right) \tag{23}$$

■

4. Conclusion

In this paper, we study the asymptotic behavior of a solution to a stochastic differential equation driven by a fractional Brownian motion (fBm). Our work is inspired by articles addressing homogenization and large deviation principle Baldi *and al.*(1991), Coulibay *and al.*(2019), Diatta *and al.*(2022). However, given the significant irregularities of fBm in $\mathbb{R}$ for $H < \frac{1}{2}$, we conduct this analysis in the space of tempered distributions $S'(\mathbb{R})$, where the fBm becomes differentiable and integral in the white noise sens Siska *and al.*(2004).

Despite the technical challenges, particularly those related to change of measure, which we have overcome, we restrict our current study to $S'(\mathbb{R})$. Nevertheless, we plan to extend this work to some infinite-dimensional spaces, as well as to other stochastic differential equations driven by fBm.

Acknowledgements. We would like to thank the referees and the editor for their substantial comments and suggestions, which have significantly improved the paper.

References

Baldi P., *Large deviations for diffusion processes with homogenization applications*, Annals of Probability. vol (19), 509–524, (1991).

Bender, C. (2003) . *An Ito formula for generalized functional san motion with arbitrary Hurst parameter, Stochastic Process. Appl. 104 , no. 1, 81-106. MR 2003m:60137.*

Coulibay A. & Diedhiou A & Sane I, *On a Skorokhod problem with small double limit*, Afrika mathematika,(**30**),5–9, (2019).

Dembo, A., Zeitouni, O., *Large deviation thechnic and applications*, second ed; Springer-verlage, New York, (1998).

Diatta, R., Diedhiou, A. (2020). *Large deviation principle applied for a solution of mixed stochastic differential equation involving independent standart Brownian motion and Fractional Brownian motion, Applied Mathematical Sciences, vol, 14, no 11, 511-530.*

Diatta, R., Diedhiou, A. (2024).*Large deviation for several fractional Brownian motion and diffusion process, International Journal of Numerical Methods and Applications 2024 Pushpa Publishing House, Prayagraj, India.*

Diatta, R., Manga, C.,*Large deviation principle for a mixed fractional Brownian motion and jump diffusion process*, Random Oper.Stoch.Equ.,Vol.30, Number 3,pp:1-9, (2022).

Diedhiou, A., Manga (2008).*Application of homogenization an large deviation to a parabolic semilinear equation, Journal Mathematical Analysis and Applications.*

Diedhiou, A., Pardoux, E. (2007).*Homogenization of semilinear hypoelliptic PDEs, Annales de la Faculté des Sciences de Taulouse. Vol XVI n 2, 2007 pp. 253-283*

Freidlin M.I., & Sowers R.B., *A comparison of homogenization and large deviations, with applications to wavefront propagation,* Stochastic Processes Appl., vol (82), 23–32, (1999).

Pardoux, E., *Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients: a probabilistic approach*, Journal of Functional Analysis 167, 498-520 (1999).

Siska, D., *Stochastic differential equations driven by fractional Brownian motion - a white noise distribution*, theory approach , (2004).

Stroock D.W., *Diffusion semigroups corresponding to uniformly elliptic divergence form operators*, Séminaire de probabilités (Strasbourg), tome 22 (1988), p. 316-347.