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Cubic transmuted Weibull distributions: formulations, unified approach, properties and empirical comparison

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Abstract. In this paper, we are interested in the cubic transmuted Weibull (*CTW*) distributions. Among the six most used cubic transmutation (*CT*) formulas developed in the literature over the years, only four have been applied to the Weibull distribution so far, and the resulting *CTW* distributions have yet to be compared. (see page 4090 for the full abstract).

Key words: cubic transmutation; Weibull distribution; parameter estimation; maximum likelihood; numerical optimization.
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Full Abtract. In this paper, we are interested in the cubic transmuted Weibull (*CTW*) distributions. Among the six most used cubic transmutation (*CT*) formulas developed in the literature over the years, only four have been applied to the Weibull distribution so far, and the resulting *CTW* distributions have yet to be compared. The contribution of this paper is twofold. First, we propose a unified formulation for *CTW* distributions and derive key statistical properties, including reliability and hazard functions, as well as moments. This unified formula and its related results account for not only existing *CTW* distributions but also all potential forms of *CTW* distributions that might be developed in the future. We apply our unified formula to derive two new *CTW* distributions based on the two remaining *CT* formulas not yet utilized. Secondly, we compare the six *CTW* distributions from an empirical perspective using real datasets and likelihood-based model selection criteria. Our model comparison suggests that all forms of *CTW* distributions are relevant for data modelling.

Résumé. (Full abstract in French) Dans cet article, nous nous intéressons aux transmutations cubiques de la loi de Weibull (*CTW*). Parmi les six formules de transmutation cubique (*CT*) les plus couramment utilisées dans la littérature au fil des années, seules quatre ont été appliquées à la distribution de Weibull, et les versions *CTW* obtenues n'ont pas encore fait l'objet d'une comparaison systématique. La contribution de ce travail est double. D'une part, nous proposons une formulation unifiée des distributions *CTW* , à partir de laquelle nous dérivons les principales propriétés statistiques, notamment les fonctions de fiabilité et de risque, ainsi que les moments. Cette approche unifiée englobe non seulement les *CTW* existantes, mais permet également de couvrir toutes les formes potentielles susceptibles d'être développées à l'avenir. Nous l'utilisons pour introduire deux nouvelles distributions *CTW* , fondées sur les deux formules de *CT* jusqu'alors inexploitées. D'autre part, nous procédons à une comparaison empirique des six distributions *CTW* à partir de jeux de données réelles, en nous appuyant sur des critères de sélection de modèles basés sur la vraisemblance. Les résultats de cette analyse suggèrent que l'ensemble des formes de *CTW* présente un intérêt pour la modélisation des données.

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1. Introduction

In modern statistics, the increasing complexity of data has motivated the development of more flexible probability distributions which can capture diverse patterns beyond the scope of classical distributions (see Ahmad et al. (2019), Ahmad et al. (2022), Khadim et al. (2022), Rahman *et al.* (2020) for detailed states of the art). An effective approach to enhance the flexibility of baseline distributions is the transmutation technique of order k ($k \in \mathbb{N}^*$), which transforms a baseline cumulative distribution function (*cdf*) $G(x)$ into a new *cdf*

$$F(x) = \sum_{i=1}^{k} \delta_i \left[G(x)\right]^i + \left(1 - \sum_{i=1}^{k} \delta_i\right) G^{k+1}(x), \tag{1}$$

where $\delta_1, \ldots, \delta_k$ are additional real parameters such that

$$\inf_{t\in[0,1]} \left[\sum_{i=1}^{k} i\delta_i t^{i-1} + (k+1)\left(1 - \sum_{i=1}^{k} \delta_i\right) t^k\right] \geqslant 0. \tag{2}$$

The first form of transmutation is the quadratic transmutation (corresponding to $k = 1$) introduced by Shaw and Buckley (2009) under the new *cdf*

$$F(x) = (1 + \lambda)G(x) - \lambda G^2(x),$$

where the condition (2) is equivalent to $|\lambda| \leqslant 1$. In the very recent years, transmutation of order 2 (usually called cubic transmutation) has gained considerable attention for its ability to increase model flexibility and improve data fitting in various applied fields such as reliability, engineering, and biology. However, cubic transmutation does not have a unique form. For example, Geraldo et al. (2024) emphasized the growing diversity of *CT* approaches, studied six existing *CT* formulas and proposed new parameter ranges to improve model performance. All the six different *CT* formulas proposed in the literature so far obey the general definition (1) under condition (2) for $k = 2$. This suggests that instead of separately developing all cubic transmutations of a baseline distribution and their properties, a general formula can be proposed. This idea was first applied by Katchekpele et al. (2025), who developed a unified formula for the cubic transmutations of the Pareto distribution, facilitating the derivation of new cubic transmuted Pareto distributions and demonstrating their effectiveness on real-world datasets.

The development of flexible probability models plays a key role in areas such as lifetime analysis, reliability engineering, actuarial science, and risk management (Lawless (2003), Murthy et al. (2004)). In this context, *CTW* distributions contribute to such developments by providing additional flexibility.

While several cubic transmutation formulas have been introduced in the literature (see, for example, Geraldo et al. (2024)), only a limited subset has been applied to the Weibull distribution. Furthermore, no unified approach exists to systematically compare their statistical properties and empirical performance. This gap restricts both theoretical understanding and practical usage of *CTW* distributions. Our work addresses this limitation by proposing a unified formulation for *CTW* distributions, deriving their key properties, and performing an empirical comparison to highlight their relative strengths.

In this paper, we are interested in the cubic transmutations of the Weibull distribution (Weibull (1951)), also called extreme value distribution of type III. This latter is defined by its *cdf*

$$G(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{\mu}\right], \quad x > 0,$$

and its probability density function (*pdf*)

$$g(x) = \frac{\mu}{\beta}\left(\frac{x}{\beta}\right)^{\mu-1} \exp\left[-\left(\frac{x}{\beta}\right)^{\mu}\right], \quad x > 0,$$

where $\beta > 0$ and $\mu > 0$ are respectively the scale and the shape parameters. The Weibull distribution is equivalent to the exponential distribution when $\mu = 1$ and to the Rayleigh distribution when $\mu = 2$. It is widely used in survival and reliability analyses and has been the focus of significant advances through cubic transmutations. Various forms of *CTW* distributions (AL-Kadim and Mohammed (2017), Aslam et al. (2020), Granzotto et al. (2017), Rahman et al. (2019a)) have enriched the Weibull family with improved statistical properties and estimation procedures. Maximum likelihood estimation (*MLE*) has been consistently employed to estimate the parameters efficiently and robustly, even in challenging data scenarios (Aslam et al. (2020)). Comparative studies also underline that cubic transmuted Weibull models often outperform baseline and quadratic transmuted Weibull distributions in fitting complex datasets.

Despite these promising developments, a unified treatment and a systematic comparison of the different cubic transmutation formulations for the Weibull distribution remain lacking. This paper aims to fill this gap by presenting a general formula that encompasses existing cubic transmutations of the Weibull distribution, providing explicit expressions for reliability and hazard functions, moments, and a detailed *MLE* framework. Through real data analysis, we assess and compare the performance of these models, highlighting their relative advantages and practical applicability. The findings in this paper account for not only existing *CTW* distributions but also all potential forms of *CTW* distributions that might be developed in the future. They thus simplify the process of deriving the properties of various *CTW* models.

The rest of this paper is organized as follows. Section 2 reviews existing cubic transmutations of the Weibull distribution. Section 3 introduces a general formula that unifies them and derives key statistical properties of the proposed model, including reliability measures and moments. In Section 4, we apply our unified formula to develop two new *CTW* distributions and their respective statistical properties.

Section 5 discusses maximum likelihood estimation and the numerical procedures used. Section 6 presents a comparative analysis based on real data. Finally, Section 7 concludes the paper by summarizing the main findings.

2. A review of existing *CTW* distributions

In this section, we review existing formulations of cubic transmutations applied to the Weibull distribution. These formulations aim to enhance the flexibility of the classical Weibull model by introducing additional parameters through cubic transformations. We highlight the main ideas, differences, and limitations of each approach as they appear in the literature. In particular, we present six known cubic transmutation formulas based on various constructions of the cumulative distribution function.

2.1. General cubic transmutation formulas

Let $x \in \mathbb{R}$ and $G(x)$ be a baseline *cdf* depending on a parameter $\xi \in \mathbb{R}^d$ ($d \in \mathbb{N}^*$). A *cdf* is a cubic transmutation of $G(x)$ if it has the form (see Katchekpele et al. (2025))

$$F(x) = \delta_1 G(x) + \delta_2 G^2(x) + (1 - \delta_1 - \delta_2)G^3(x),$$

where

$$\inf_{t\in[0,1]} \left(\delta_1 + 2\delta_2 t + 3(1 - \delta_1 - \delta_2)t^2\right) \geqslant 0. \tag{3}$$

In the following, we present a brief review (see Geraldo et al. (2024) for a detailed review), adopting the same notations as in Geraldo et al. (2024). The first cubic transmutation formula (denoted CT_G), due to Granzotto et al. (2017), corresponds to the new *cdf*

$$F_G(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1)G^2(x) + (1 - \lambda_2)G^3(x),$$

where $(\lambda_1, \lambda_2) \in \mathcal{S}_G = [0, 1] \times [-1, 1]$. The second formula, proposed by AL-Kadim and Mohammed (2017) (that we will denote CT_A), corresponds to the new *cdf*

$$F_A(x) = (1 + \lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x),$$

where $\lambda \in \mathcal{S}_A = [-1, 1]$. The third formula, proposed by Rahman et al. (2018a), corresponds to the *cdf*

$$F_{R18a}(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x),$$

where

$$(\lambda_1, \lambda_2) \in \mathcal{S}_{R18a} = \left\{(\lambda_1, \lambda_2) \in [-1, 1]^2, \quad -2 \leqslant \lambda_1 + \lambda_2 \leqslant 1\right\}.$$

The fourth formula, proposed by Rahman et al. (2018b), corresponds to the *cdf*

$$F_{R18b}(x) = (1 + \lambda_1 + \lambda_2)G(x) - (\lambda_1 + 2\lambda_2)G^2(x) + \lambda_2 G^3(x),$$

where $(\lambda_1, \lambda_2) \in \mathcal{S}_{R18b} = [-1, 1] \times [0, 1]$. The fifth formula, proposed by Rahman et al. (2019b), corresponds to the *cdf*

$$F_{R19}(x) = (1-\lambda)G(x) + 3\lambda G^2(x) - 2\lambda G^3(x),$$

where $\lambda \in \mathcal{S}_{R19} = [-1, 1]$. And the sixth (and last) formula, proposed by Rahman et al. (2023), corresponds to the *cdf*

$$F_{R23}(x) = [1-\lambda(\theta-1)]\, G(x) + \lambda(2\theta-1)G^2(x) - \lambda\theta G^3(x),$$

where $(\lambda, \theta) \in \mathcal{S}_{R23} = [-1, 1] \times [0, 2]$.

2.2. Improvements in cubic transmutation formulas

Building on the formulations presented above, further developments have aimed to address some of the limitations identified in the mathematical formulations of cubic transmutation models and enhance their flexibility and applicability. Among these, an important contribution is due to Geraldo et al. (2024), who investigated the admissible ranges of the transmutation parameters to ensure well-defined *cdfs*. Focusing on the six main cubic transmutation formulas, the authors proposed extended parameter ranges for five of them: CT_G, CT_A, CT_{R18a}, CT_{R18b}, and CT_{R19}. Their respective extended ranges are given by

$$\mathcal{S}_{MG} = \Big\{(\lambda_1, \lambda_2) \in [0, 3] \times [0, 3], \quad 0 \leqslant \lambda_1 + \lambda_2 \leqslant 3\Big\},$$

$$\mathcal{S}_{MA} = [-1, 3],$$

$$\mathcal{S}_{MR18a} = \Big\{(\lambda_1, \lambda_2) \in [-1, 2] \times [-1, 2], \quad -2 \leqslant \lambda_1 + \lambda_2 \leqslant 1\Big\},$$

$$\mathcal{S}_{MR18b} = \Big\{(\lambda_1, \lambda_2) \in [-2, 1] \times [-2, 1], \quad -1 \leqslant \lambda_1 + \lambda_2 \leqslant 2\Big\}$$

and

$$\mathcal{S}_{MR19} = [-2, 1].$$

The modified versions under these extended ranges were denoted CT_{MG}, CT_{MA}, CT_{MR18a}, CT_{MR18b}, and CT_{MR19}, respectively. Their analysis ensured that each of the resulting cumulative distribution functions remains valid across the new parameter domains. To validate the practical relevance of these enhancements, the authors conducted a comparative study using the Pareto distribution as a baseline. Their results indicated that the modified transmutation formulas perform at least as well as the original ones in fitting real data. It is worth noting that no such extension of the parameter range has been proposed for the CT_{R23} model. Geraldo et al. (2024) then proved that for all $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$, the $CT_{MG}(\lambda_1, \lambda_2)$, the $CT_{MR18a}(\lambda_1 - 1, \lambda_2 - 1)$ and the $CT_{MR18b}(\lambda_1 + \lambda_2 - 2, 1 - \lambda_2)$ families of distributions linked to $G(x)$ are equivalent.

2.3. Existing CTW distributions

Using the quadratic transmutation proposed by Shaw and Buckley (2009), Aryal and Tsokos (2011) developed the transmuted Weibull (*TW*) distribution. Their work was extended by Khan et al. (2017) who studied further properties of the *TW* distribution. Several authors have been interested in the *CT* of Weibull (*CTW*) distribution. Granzotto et al. (2017) and AL-Kadim and Mohammed (2017) applied their own formulas CT_G and CT_A to derive two *CTW* distributions respectively denoted CTW_G and CTW_A. Rahman et al. (2019a) proposed a *CTW* distribution using the formula CTW_{R18a}. The transmutation formula CT_{R19} has also been considered by Aslam et al. (2020) and applied to Weibull distribution to obtain another *CTW* hereafter denoted CTW_{R19}. To the best of our knowledge, the CT_{R18b} and CT_{R23} have not been applied to the Weibull distribution.

3. Unified formula for the *CTW* distribution

In this section, we present the general form of the *cdf* and *pdf* for the *CTW* distribution, derived from the unified cubic transmutation applied to the Weibull baseline.

3.1. Cumulative distribution and probability density functions

Proposition 1. *The cdf and pdf of any CTW distribution are respectively given by*

$$F(x) = 1 + (2\delta_1 + \delta_2 - 3)e^{-(x/\beta)^\mu} + (3 - 3\delta_1 - 2\delta_2)e^{-2(x/\beta)^\mu} + (\delta_1 + \delta_2 - 1)e^{-3(x/\beta)^\mu} \quad (4)$$

and

$$f(x) = \frac{\mu}{\beta^\mu}x^{\mu-1}\Big[(3 - 2\delta_1 - \delta_2)e^{-(x/\beta)^\mu} + (6\delta_1 + 4\delta_2 - 6)e^{-2(x/\beta)^\mu} + (3 - 3\delta_1 - 3\delta_2)e^{-3(x/\beta)^\mu}\Big], \quad (5)$$

where $\beta > 0$, $x > 0$, $\mu > 0$ *and* δ_1 *and* δ_2 *satisfy the conditions* (3) *ensuring that* $F(x)$ *is a well-defined cdf.*

Proof. Let $F(x)$ and $f(x)$ respectively be the *cdf* and *pdf* of the *CTW* distribution. We have $f(x) = F'(x)$. Now,

$$\begin{aligned} F(x) &= \delta_1\left(1 - e^{-(x/\beta)^\mu}\right) + \delta_2\left(1 - e^{-(x/\beta)^\mu}\right)^2 + (1 - \delta_1 - \delta_2)\left(1 - e^{-(x/\beta)^\mu}\right)^3 \\ &= \delta_1\left(1 - e^{-(x/\beta)^\mu}\right) + \delta_2\left(1 - 2e^{-(x/\beta)^\mu} + e^{-2(x/\beta)^\mu}\right) \\ &\quad + (1 - \delta_1 - \delta_2)\left(1 - 3e^{-(x/\beta)^\mu} + 3e^{-2(x/\beta)^\mu} - e^{-3(x/\beta)^\mu}\right) \\ &= 1 + (2\delta_1 + \delta_2 - 3)e^{-(x/\beta)^\mu} + (3 - 3\delta_1 - 2\delta_2)e^{-2(x/\beta)^\mu} + (\delta_1 + \delta_2 - 1)e^{-3(x/\beta)^\mu}. \end{aligned}$$

Thus,

$$f(x) = \frac{\mu}{\beta^\mu} x^{\mu-1} \Big[(3 - 2\delta_1 - \delta_2) e^{-(x/\beta)^\mu} + (6\delta_1 + 4\delta_2 - 6) e^{-2(x/\beta)^\mu} + (3 - 3\delta_1 - 3\delta_2) e^{-3(x/\beta)^\mu} \Big].$$

Which completes the proof. □

Remark 1. Applying formulas (4) and (5) for specified values of δ_1 and δ_2 enables us to recover the *cdf* and *pdf* of the existing *CTW* distributions.

- Setting $\delta_1 = \lambda_1$ and $\delta_2 = \lambda_2 - \lambda_1$ enables us to recover the *cdf* and the *pdf* of the CTW_G distribution of Granzotto et al. (2017):

$$F_G(x) = 1 + (\lambda_1 + \lambda_2 - 3) e^{-(x/\beta)^\mu} + (3 - \lambda_1 - 2\lambda_2) e^{-2(x/\beta)^\mu} + (\lambda_2 - 1) e^{-3(x/\beta)^\mu}$$

  and

$$f_G(x) = \frac{\mu}{\beta^\mu} x^{\mu-1} \Big[(3 - \lambda_1 - \lambda_2) e^{-(x/\beta)^\mu} + 2(\lambda_1 + 2\lambda_2 - 3) e^{-2(x/\beta)^\mu} + 3(1 - \lambda_2) e^{-3(x/\beta)^\mu} \Big].$$

- Setting $\delta_1 = 1 + \lambda$ and $\delta_2 = -2\lambda$ enables us to recover the *cdf* and *pdf* of the CTW_A distribution of AL-Kadim and Mohammed (2017):

$$F_A(x) = 1 - e^{-(x/\beta)^\mu} + \lambda e^{-2(x/\beta)^\mu} - \lambda e^{-3(x/\beta)^\mu}$$

  and

$$f_A(x) = \frac{\mu}{\beta^\mu} x^{\mu-1} \Big[e^{-(x/\beta)^\mu} - 2\lambda e^{-2(x/\beta)^\mu} + 3\lambda e^{-3(x/\beta)^\mu} \Big].$$

- Setting $\delta_1 = 1 + \lambda_1$ and $\delta_2 = \lambda_2 - \lambda_1$ enables us to recover the *cdf* and *pdf* of the CTW_{R18a} distribution of Rahman et al. (2019a) based on the CT_{R18a} formula:

$$F_{R18a}(x) = 1 + (\lambda_1 + \lambda_2 - 1) e^{-(x/\beta)^\mu} - (\lambda_1 + 2\lambda_2) e^{-2(x/\beta)^\mu} + \lambda_2 e^{-3(x/\beta)^\mu}$$

  and

$$f_{R18a}(x) = \frac{\mu}{\beta^\mu} x^{\mu-1} \Big[(1 - \lambda_1 - \lambda_2) e^{-(x/\beta)^\mu} + 2(\lambda_1 + 2\lambda_2) e^{-2(x/\beta)^\mu} - 3\lambda_2 e^{-3(x/\beta)^\mu} \Big].$$

- Setting $\delta_1 = 1 - \lambda$ and $\delta_2 = 3\lambda$ enables us to recover the *cdf* and *pdf* of the CTW_{R19} distribution of Aslam et al. (2020) based on the CT_{R19} formula:

$$F_{R19}(x) = 1 + (\lambda - 1) e^{-(x/\beta)^\mu} - 3\lambda e^{-2(x/\beta)^\mu} + 2\lambda e^{-3(x/\beta)^\mu}$$

  and

$$f_{R19}(x) = \frac{\mu}{\beta^\mu} x^{\mu-1} \Big[(1 - \lambda) e^{-(x/\beta)^\mu} + 6\lambda e^{-2(x/\beta)^\mu} - 6\lambda e^{-3(x/\beta)^\mu} \Big].$$

3.2. Reliability and hazard functions

Proposition 2. *The reliability and hazard functions of the CTW distribution are respectively given for all $t > 0$ by*

$$\overline{F}(t) = (3-2\delta_1-\delta_2)e^{-(t/\beta)^\mu} + (3\delta_1+2\delta_2-3)e^{-2(t/\beta)^\mu} + (1-\delta_1-\delta_2)e^{-3(t/\beta)^\mu}$$

and

$$h(t) = \frac{\mu}{\beta^\mu}t^{\mu-1}\left[\frac{(3-2\delta_1-\delta_2)+(6\delta_1+4\delta_2-6)e^{-(t/\beta)^\mu}+(3-3\delta_1-3\delta_2)e^{-2(t/\beta)^\mu}}{(3-2\delta_1-\delta_2)+(3\delta_1+2\delta_2-3)e^{-(t/\beta)^\mu}+(1-\delta_1-\delta_2)e^{-2(t/\beta)^\mu}}\right].$$

Proof. The proof is straightforward, recalling that for all $t > 0$, the reliability function is given by $\overline{F}(t) = 1-F(t)$, and the hazard function is defined by $h(t) = f(t)/\overline{F}(t)$. □

Setting the values of δ_1 and δ_2 as in Remark 1, enables us to recover the reliability and hazard functions associated with the existing four *CTW* distributions.

3.3. Moments

Proposition 3. *Let X be a random variable following the CTW distribution with pdf (5). For all $k \in \mathbb{N}^*$, the k^{th} moment is given by:*

$$\mathbb{E}(X^k) = \beta^k\Gamma\left(\frac{\mu+k}{\mu}\right)\left[(3-2\delta_1-\delta_2)+\frac{(3\delta_1+2\delta_2-3)}{2^{k/\mu}}+\frac{(1-\delta_1-\delta_2)}{3^{k/\mu}}\right], \tag{6}$$

where Γ is the Gamma function. The mean and variance are respectively given by

$$\mathbb{E}(X) = \beta\Gamma\left(\frac{\mu+1}{\mu}\right)\left[(3-2\delta_1-\delta_2)+\frac{(3\delta_1+2\delta_2-3)}{2^{1/\mu}}+\frac{(1-\delta_1-\delta_2)}{3^{1/\mu}}\right]$$

and

$$\begin{aligned}\mathrm{Var}(X) = {} & \beta^2\Gamma\left(\frac{\mu+2}{\mu}\right)\left[(3-2\delta_1-\delta_2)+\frac{(3\delta_1+2\delta_2-3)}{2^{2/\mu}}+\frac{(1-\delta_1-\delta_2)}{3^{2/\mu}}\right] \\ & -\left(\beta\Gamma\left(\frac{\mu+1}{\mu}\right)\left[(3-2\delta_1-\delta_2)+\frac{(3\delta_1+2\delta_2-3)}{2^{1/\mu}}+\frac{(1-\delta_1-\delta_2)}{3^{1/\mu}}\right]\right)^2.\end{aligned}$$

Proof. By definition, the k^{th} moment is

$$\mathbb{E}(X^k) = \int_{-\infty}^{\infty} x^k f(x)\,\mathrm{d}x.$$

Applying this formula to the *pdf* $f(x)$ defined by formula (5), we get:

$$
\begin{aligned}
\mathbb{E}(X^k) = & \int_0^\infty x^k \frac{\mu}{\beta^\mu} x^{\mu-1} \Big[(3-2\delta_1-\delta_2) e^{-(x/\beta)^\mu} + (6\delta_1+4\delta_2-6) e^{-2(x/\beta)^\mu} \\
& \qquad + (3-3\delta_1-3\delta_2) e^{-3(x/\beta)^\mu} \Big] \, \mathrm{d}x \\
= & (3-2\delta_1-\delta_2) \int_0^\infty \frac{\mu}{\beta^\mu} x^{\mu+k-1} e^{-(x/\beta)^\mu} \, \mathrm{d}x \\
& + (6\delta_1+4\delta_2-6) \int_0^\infty \frac{\mu}{\beta^\mu} x^{\mu+k-1} e^{-2(x/\beta)^\mu} \, \mathrm{d}x \\
& + (3-3\delta_1-3\delta_2) \int_0^\infty \frac{\mu}{\beta^\mu} x^{\mu+k-1} e^{-3(x/\beta)^\mu} \, \mathrm{d}x.
\end{aligned}
$$

By setting respectively $t = (x/\beta)^\mu$, $t = 2(x/\beta)^\mu$ and $t = 3(x/\beta)^\mu$, we have:

$$
\begin{aligned}
\mathbb{E}(X^k) = & (3-2\delta_1-\delta_2) \int_0^\infty (\beta t^{1/\mu})^k e^{-t} \, \mathrm{d}t + (6\delta_1+4\delta_2-6) \int_0^\infty \frac{1}{2} \Big(\beta\Big(\frac{t}{2}\Big)^{1/\mu}\Big)^k e^{-t} \, \mathrm{d}t \\
& + (3-3\delta_1-3\delta_2) \int_0^\infty \frac{1}{3} \Big(\beta\Big(\frac{t}{3}\Big)^{1/\mu}\Big)^k e^{-t} \, \mathrm{d}t \\
= & (3-2\delta_1-\delta_2) \int_0^\infty \beta^k t^{k/\mu} e^{-t} \, \mathrm{d}t + (6\delta_1+4\delta_2-6) \int_0^\infty \frac{\beta^k}{2^{(k/\mu)+1}} t^{k/\mu} e^{-t} \, \mathrm{d}t \\
& + (3-3\delta_1-3\delta_2) \int_0^\infty \frac{\beta^k}{3^{(k/\mu)+1}} t^{k/\mu} e^{-t} \, \mathrm{d}t \\
= & (3-2\delta_1-\delta_2) \beta^k \Gamma\left(\frac{k}{\mu}+1\right) + (6\delta_1+4\delta_2-6) \frac{\beta^k}{2^{(k/\mu)+1}} \Gamma\left(\frac{k}{\mu}+1\right) \\
& + (3-3\delta_1-3\delta_2) \frac{\beta^k}{3^{(k/\mu)+1}} \Gamma\left(\frac{k}{\mu}+1\right) \\
= & \beta^k \Gamma\left(\frac{\mu+k}{\mu}\right) \left[(3-2\delta_1-\delta_2) + \frac{(6\delta_1+4\delta_2-6)}{2^{(k/\mu)+1}} + \frac{(3-3\delta_1-3\delta_2)}{3^{(k/\mu)+1}} \right] \\
= & \beta^k \Gamma\left(\frac{\mu+k}{\mu}\right) \left[(3-2\delta_1-\delta_2) + \frac{(3\delta_1+2\delta_2-3)}{2^{k/\mu}} + \frac{(1-\delta_1-\delta_2)}{3^{k/\mu}} \right].
\end{aligned}
$$

The mathematical expectation and variance can be easily derived by taking $k = 1$, $k = 2$, and using the relation $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$. □

Remark 2. By applying formula (6) for specified values of δ_1 and δ_2, one recovers the moments of two of the four existing *CTW* distributions (the two others being either erroneous or unspecified).

- Setting $\delta_1 = 1 + \lambda$ and $\delta_2 = -2\lambda$ enables one to recover the k^{th} moment of the CTW_A distribution:

  $$\mathbb{E}(X^k) = \beta^k \Gamma\left(\frac{\mu+k}{\mu}\right) \left[1 - \frac{\lambda}{2^{k/\mu}} + \frac{\lambda}{3^{k/\mu}} \right].$$

  The result is identical to that found in AL-Kadim and Mohammed (2017), Proposition 1, p. 869 when considering the notation of these authors.

- Setting $\delta_1 = 1 + \lambda_1$ and $\delta_2 = \lambda_2 - \lambda_1$ enables one to recover the k^{th} moment of the CTW_{R18a} distribution:

$$\mathbb{E}(X^k) = \beta^k \Gamma\left(\frac{\mu + k}{\mu}\right)\left[(1 - \lambda_1 - \lambda_2) + \frac{(\lambda_1 + 2\lambda_2)}{2^{k/\mu}} - \frac{\lambda_2}{3^{k/\mu}}\right].$$

  Here also, the result is identical to that found in Rahman et al. (2019a), Theorem 1.

The following result, obtained by setting $\delta_1 = \lambda_1$ and $\delta_2 = \lambda_2 - \lambda_1$ in formula (6), is new and rectifies an error present in Granzotto et al. (2017), p. 2764.

Corollary 1. *The k^{th} moment of the CTW_G distribution of Granzotto et al. (2017) is given by:*

$$\mathbb{E}(X^k) = \beta^k \Gamma\left(\frac{\mu + k}{\mu}\right)\left[(3 - \lambda_1 - \lambda_2) + \frac{(\lambda_1 + 2\lambda_2 - 3)}{2^{k/\mu}} + \frac{(1 - \lambda_2)}{3^{k/\mu}}\right].$$

The following result, obtained by setting $\delta_1 = 1 - \lambda$ and $\delta_2 = 3\lambda$ in formula (6), is also new because the moments of the CTW_{R19} distribution were not calculated explicitly in Aslam et al. (2020).

Corollary 2. *The k^{th} moment of the CTW_{R19} distribution of Aslam et al. (2020) is given by:*

$$\mathbb{E}(X^k) = \beta^k \Gamma\left(\frac{\mu + k}{\mu}\right)\left[(1 - \lambda) + \frac{3\lambda}{2^{k/\mu}} - \frac{\lambda}{3^{k/\mu}}\right].$$

4. Two new *CTW* distributions

As mentioned above, only four of the six existing CT formulas have been applied so far to the Weibull distribution. We can now apply formulas (4) and (5) of Proposition 1 to derive two new corollaries giving the respective *cdf* and *pdf* linked to the remaining two CT formulas i.e. CT_{R18b} and CT_{R23}.

4.1. First new CTW distribution

Setting $\delta_1 = 1 + \lambda_1 + \lambda_2$ and $\delta_2 = -\lambda_1 - 2\lambda_2$ in formulas (4) and (5), we obtain a new *CTW* denoted CTW_{R18b} based on the application of the CT_{R18b} formula to the Weibull distribution. The *cdf* and *pdf* of the new CTW_{R18b} distribution are respectively given by

$$F_{R18b}(x) = 1 + (\lambda_1 - 1)e^{-(x/\beta)^\mu} + (\lambda_2 - \lambda_1)e^{-2(x/\beta)^\mu} - \lambda_2 e^{-3(x/\beta)^\mu} \tag{7}$$

and

$$f_{R18b}(x) = \frac{\mu}{\beta^\mu} x^{\mu-1}\left[(1 - \lambda_1)e^{-(x/\beta)^\mu} + 2(\lambda_1 - \lambda_2)e^{-2(x/\beta)^\mu} + 3\lambda_2 e^{-3(x/\beta)^\mu}\right].$$

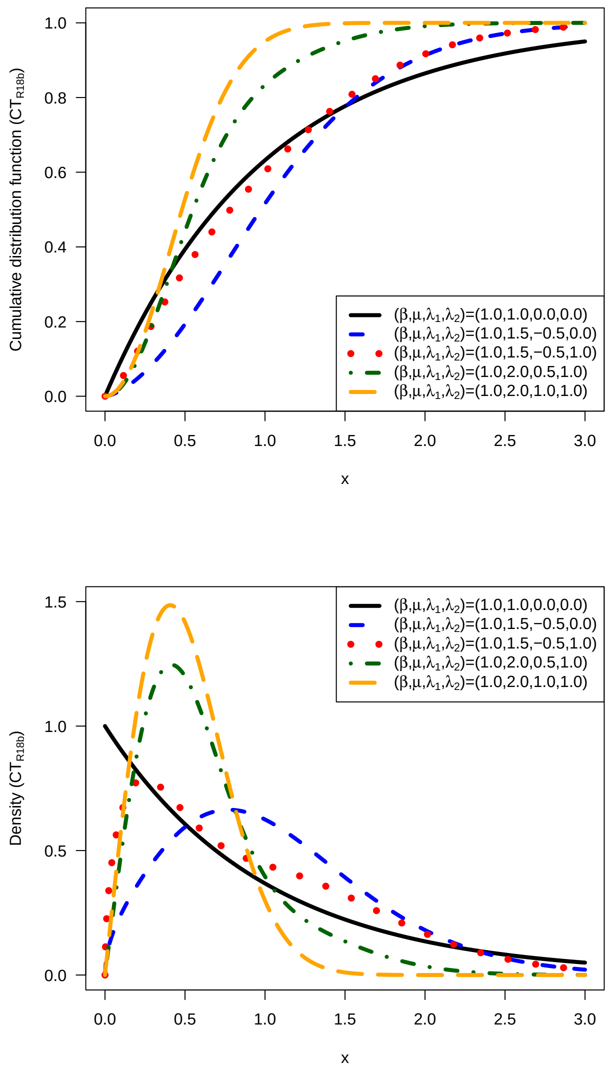


Fig. 1. Cumulative distribution function and probability density function of the distribution CTW_{R18b} for specified parameter values, illustrating the effect of the transmutation parameters on the distribution shape.

Figure 1 gives the shape of the *cdf* and *pdf* of the CTW_{R18b} distribution for specified values of its parameters.
By setting $\delta_1 = 1+\lambda_1+\lambda_2$ and $\delta_2 = -\lambda_1-2\lambda_2$ in Proposition 2, we obtain the reliability and hazard rate functions of the new CTW_{R18b} distribution as

$$\overline{F}_{R18b}(t) = (1-\lambda_1)e^{-(t/\beta)^\mu} + (\lambda_1-\lambda_2)e^{-2(t/\beta)^\mu} + \lambda_2 e^{-3(t/\beta)^\mu}$$

and

$$h_{R18b}(t) = \frac{\mu}{\beta^\mu} t^{\mu-1} \left[\frac{(1-\lambda_1) + 2(\lambda_1-\lambda_2)e^{-(t/\beta)^\mu} + 3\lambda_2 e^{-2(t/\beta)^\mu}}{(1-\lambda_1) + (\lambda_1-\lambda_2)e^{-(t/\beta)^\mu} + \lambda_2 e^{-2(t/\beta)^\mu}} \right].$$

Figure 2 gives the shape of the reliability and hazard rate functions of the CTW_{R18b} distribution for specified values of its parameters.
The following result, obtained by setting respectively $\delta_1 = 1 + \lambda_1 + \lambda_2$ and $\delta_2 = -\lambda_1 - 2\lambda_2$ in formula (6) is also new and concerns the mathematical expectation of the CTW_{R18b} distribution.

Corollary 3. *The k^{th} moment of the CTW_{R18b} distribution is given by:*

$$\mathbb{E}(X^k) = \beta^k \Gamma\left(\frac{\mu+k}{\mu}\right) \left[(1-\lambda_1) + \frac{(\lambda_1-\lambda_2)}{2^{k/\mu}} + \frac{\lambda_2}{3^{k/\mu}} \right].$$

4.2. Second new CTW distribution

Setting $\delta_1 = 1+\lambda-\lambda\theta$ and $\delta_2 = 2\lambda\theta-\lambda$ in formulas (4) and (5), we obtain a second new *CTW* denoted CTW_{R23} based on the application of the CT_{R23} formula to the Weibull distribution. The *cdf* and *pdf* of the new CTW_{R23} distribution are respectively given by

$$F_{R23}(x) = 1 + (\lambda-1)e^{-(x/\beta)^\mu} - \lambda(1+\theta)e^{-2(x/\beta)^\mu} + \lambda\theta e^{-3(x/\beta)^\mu}$$

and

$$f_{R23}(x) = \frac{\mu}{\beta^\mu} x^{\mu-1} \left[(1-\lambda)e^{-(x/\beta)^\mu} + 2\lambda(1+\theta)e^{-2(x/\beta)^\mu} - 3\lambda\theta e^{-3(x/\beta)^\mu} \right].$$

Figure 3 gives the shape of the *cdf* and *pdf* of the CTW_{R23} distribution for specified values of its parameters.
By setting $\delta_1 = 1 + \lambda - \lambda\theta$ and $\delta_2 = 2\lambda\theta - \lambda$ in Proposition 2, we obtain the *cdf* and *pdf* of the new CTW_{R23} distribution as

$$\overline{F}_{R23}(t) = (1-\lambda)e^{-(t/\beta)^\mu} + \lambda(1+\theta)e^{-2(t/\beta)^\mu} - \lambda\theta e^{-3(t/\beta)^\mu}$$

and

$$h_{R23}(t) = \frac{\mu}{\beta^\mu} t^{\mu-1} \left[\frac{(1-\lambda) + 2\lambda(1+\theta)e^{-(t/\beta)^\mu} - 3\lambda\theta e^{-2(t/\beta)^\mu}}{(1-\lambda) + \lambda(1+\theta)e^{-(t/\beta)^\mu} - \lambda\theta e^{-2(t/\beta)^\mu}} \right].$$

Figure 4 gives the shape of the reliability and hazard rate functions of the CTW_{R23} distribution for specified values of its parameters.
The following result, obtained by setting respectively $\delta_1 = 1+\lambda-\lambda\theta$ and $\delta_2 = 2\lambda\theta-\lambda$ in formula (6) is also new and concern the mathematical expectation of the CTW_{R23} distribution.

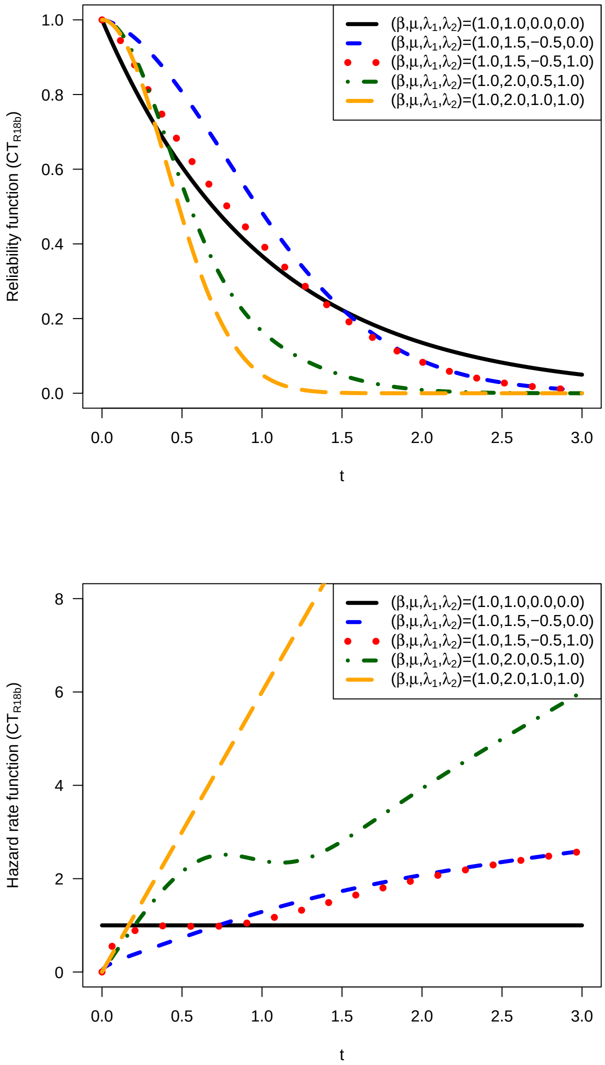
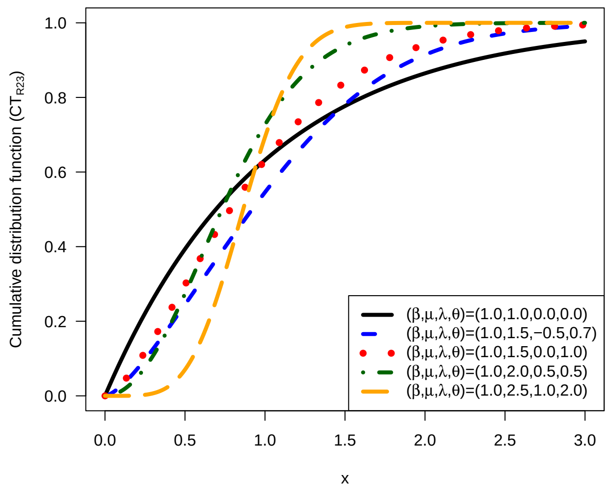


Fig. 2. Reliability function and hazard rate function of the cubic transmuted Weibull distribution CTW_{R18b} for specified parameter values, highlighting changes in survival and failure rates due to the transmutation parameters.

Corollary 4. *The k^{th} moment of the CTW_{R23} distribution is given by:*

$$\mathbb{E}(X^k) = \beta^k \Gamma\left(\frac{\mu + k}{\mu}\right) \left[(1-\lambda) + \frac{\lambda(1+\theta)}{2^{k/\mu}} - \frac{\lambda\theta}{3^{k/\mu}}\right].$$



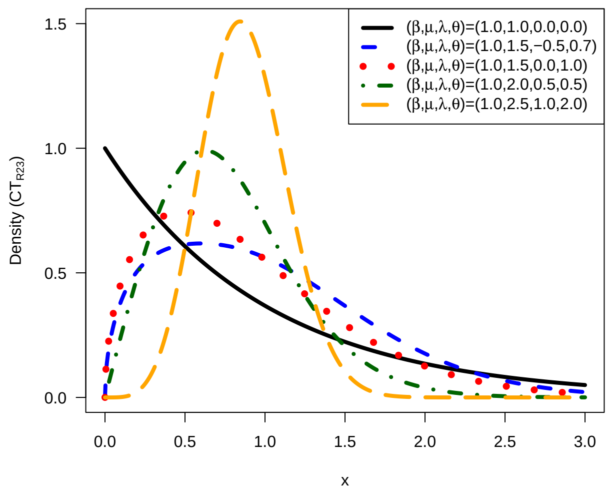


Fig. 3. Cumulative distribution function and probability density function of the cubic transmuted Weibull distribution CTW_{R23} for specified parameter values, showing how the distribution shape is affected by parameter choices.

5. Parameter estimation

This section focuses on the maximum likelihood estimation of the parameters of each model, providing the necessary log-likelihood functions and discussing the practical aspects of the estimation procedure. The aim of this paper is not to com-

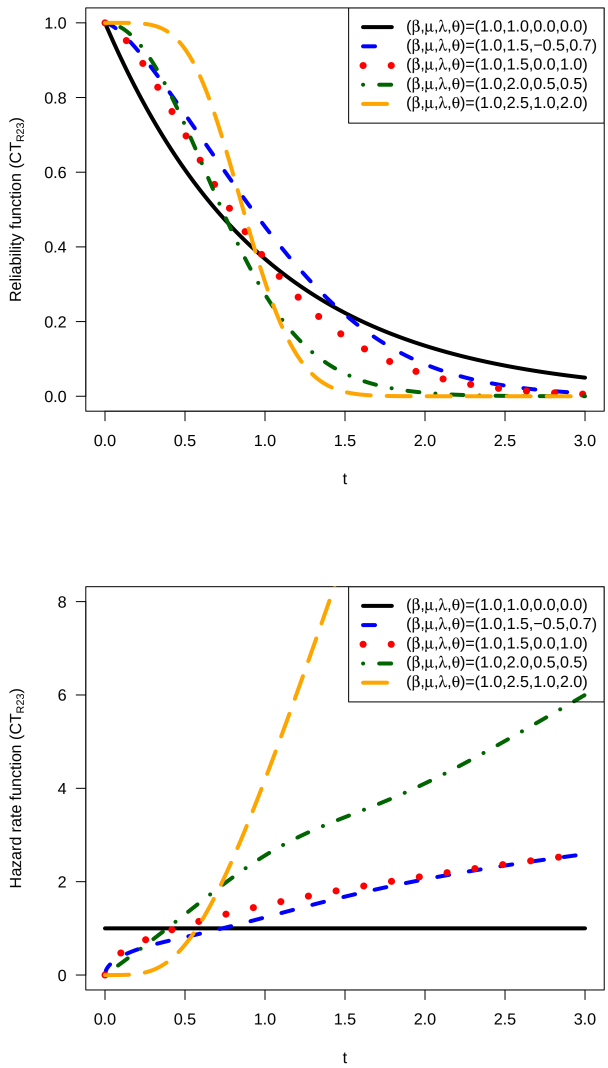


Fig. 4. Reliability function and hazard rate function of the cubic transmuted Weibull distribution CTW_{R23} for specified parameter values, illustrating the impact of the transmutation parameters on survival and hazard behavior.

pare the impact of initial parameter ranges and their modified versions. The recent research works (see, for example, Geraldo et al. (2024), Katchekpele et al. (2025)) have sufficiently proven that the cubic transmuted distributions are better in terms of data fitting with modified ranges than the initial ranges. Therefore, in the remainder of this paper, we will only consider the different *CTW* under their respective modified parameter ranges. The CTW_G, CTW_A, CTW_{R18a}, CTW_{R18b}, and CTW_{R19} distributions, under their respective modified parameter ranges $\mathcal{S}_{MG}$, $\mathcal{S}_{MA}$, $\mathcal{S}_{MR18a}$, $\mathcal{S}_{MR18b}$, and $\mathcal{S}_{MR19}$, will be respectively denoted as CTW_{MG}, CTW_{MA}, CTW_{MR18a}, CTW_{MR18b}, and CTW_{MR19}.

Let $x_1, \ldots, x_n$ be a random sample of size n from the Cubic Transmuted Weibull distribution defined by Proposition 1. The log-likelihood is given by

$$\ell = \log\left(\prod_{i=1}^{n} f(x_i)\right) = \sum_{i=1}^{n} \log f(x_i),$$

where the *pdf* f is given by formula (5). By considering the different forms of *CTW* and their respective parameters, we have the following cases.

- The log-likelihood corresponding to the CTW_{MG} distribution is

$$\begin{aligned}\ell_{MG} = n\log(\mu) - n\mu\log(\beta) + (\mu-1)\sum_{i=1}^{n}\log x_i \\ + \sum_{i=1}^{n}\log\Big[(3-\lambda_1-\lambda_2)e^{-(x_i/\beta)^{\mu}} + 2(\lambda_1+2\lambda_2-3)e^{-2(x_i/\beta)^{\mu}} \\ + 3(1-\lambda_2)e^{-3(x_i/\beta)^{\mu}}\Big],\end{aligned}$$

  and the maximum likelihood estimators (*MLEs*) $\hat{\beta}$, $\hat{\mu}$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$ of β, μ, λ_1, and λ_2 are such that

$$(\hat{\beta}, \hat{\mu}, \hat{\lambda}_1, \hat{\lambda}_2) = \underset{(\beta,\mu,\lambda_1,\lambda_2)\in\mathbb{R}_+^*\times\mathbb{R}_+^*\times\mathcal{S}_{MG}}{\arg\max} \ell_{MG}(\beta,\mu,\lambda_1,\lambda_2).$$

- The log-likelihood corresponding to the CTW_{MA} distribution is

$$\begin{aligned}\ell_{MA} = n\log(\mu) - n\mu\log(\beta) + (\mu-1)\sum_{i=1}^{n}\log x_i \\ + \sum_{i=1}^{n}\log\Big[e^{-(x_i/\beta)^{\mu}} - 2\lambda e^{-2(x_i/\beta)^{\mu}} + 3\lambda e^{-3(x_i/\beta)^{\mu}}\Big],\end{aligned}$$

  and the *MLEs* $\hat{\beta}$, $\hat{\mu}$, and $\hat{\lambda}$ of β, μ, and λ are such that

$$(\hat{\beta}, \hat{\mu}, \hat{\lambda}) = \underset{(\beta,\mu,\lambda)\in\mathbb{R}_+^*\times\mathbb{R}_+^*\times[-1,3]}{\arg\max} \ell_{MA}(\beta,\mu,\lambda).$$

- The log-likelihood corresponding to the CTW_{MR18a} distribution is

$$\ell_{MR18a} = n\log(\mu) - n\mu\log(\beta) + (\mu-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\Big[(1-\lambda_1-\lambda_2)e^{-(x_i/\beta)^{\mu}} + 2(\lambda_1+2\lambda_2)e^{-2(x_i/\beta)^{\mu}} - 3\lambda_2 e^{-3(x_i/\beta)^{\mu}}\Big],$$

and the *MLEs* $\hat{\beta}$, $\hat{\mu}$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$ of β, μ, λ_1, and λ_2 are such that

$$(\hat{\beta},\hat{\mu},\hat{\lambda}_1,\hat{\lambda}_2) = \underset{(\beta,\mu,\lambda_1,\lambda_2)\in\mathbb{R}_+^*\times\mathbb{R}_+^*\times\mathcal{S}_{MR18a}}{\arg\max}\ \ell_{MR18a}(\beta,\mu,\lambda_1,\lambda_2).$$

- The log-likelihood corresponding to the CTW_{MR18b} distribution is

$$\ell_{MR18b} = n\log(\mu) - n\mu\log(\beta) + (\mu-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\Big[(1-\lambda_1)e^{-(x_i/\beta)^{\mu}} + 2(\lambda_1-\lambda_2)e^{-2(x_i/\beta)^{\mu}} + 3\lambda_2 e^{-3(x_i/\beta)^{\mu}}\Big],$$

and the *MLEs* $\hat{\beta}$, $\hat{\mu}$, $\hat{\lambda}_1$, and $\hat{\lambda}_2$ of β, μ, λ_1, and λ_2 are such that

$$(\hat{\beta},\hat{\mu},\hat{\lambda}_1,\hat{\lambda}_2) = \underset{(\beta,\mu,\lambda_1,\lambda_2)\in\mathbb{R}_+^*\times\mathbb{R}_+^*\times\mathcal{S}_{MR18b}}{\arg\max}\ \ell_{MR18b}(\beta,\mu,\lambda_1,\lambda_2).$$

- The log-likelihood corresponding to the CTW_{MR19} distribution is

$$\ell_{MR19} = n\log(\mu) - n\mu\log(\beta) + (\mu-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\Big[(1-\lambda)e^{-(x_i/\beta)^{\mu}} + 6\lambda e^{-2(x_i/\beta)^{\mu}} - 6\lambda e^{-3(x_i/\beta)^{\mu}}\Big],$$

and the *MLEs* $\hat{\beta}$, $\hat{\mu}$, and $\hat{\lambda}$ of β, μ, and λ are such that

$$(\hat{\beta},\hat{\mu},\hat{\lambda}) = \underset{(\beta,\mu,\lambda)\in\mathbb{R}_+^*\times\mathbb{R}_+^*\times[-2,1]}{\arg\max}\ \ell_{MR19}(\beta,\mu,\lambda).$$

- The log-likelihood corresponding to the CTW_{R23} distribution is

$$\ell_{R23} = n\log(\mu) - n\mu\log(\beta) + (\mu-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log\Big[(1-\lambda)e^{-(x_i/\beta)^{\mu}} + 2\lambda(1+\theta)e^{-2(x_i/\beta)^{\mu}} - 3\lambda\theta e^{-3(x_i/\beta)^{\mu}}\Big],$$

and the *MLEs* $\hat{\beta}$, $\hat{\mu}$, $\hat{\lambda}$, and $\hat{\theta}$ of β, μ, λ, and θ are such that

$$(\hat{\beta}, \hat{\mu}, \hat{\lambda}, \hat{\theta}) = \underset{(\beta,\mu,\lambda,\theta)\in\mathbb{R}_+^*\times\mathbb{R}_+^*\times[-1,1]\times[0,2]}{\arg\max} \ell_{R23}(\beta, \mu, \lambda, \theta).$$

All these constrained optimization problems are difficult to solve analytically and therefore require numerical optimization algorithms. A key requirement for such algorithms is the ability to handle inequality constraints. In the remainder of this paper, we use the "constrOptim" function from the R software (R Core Team (2025)), which implements several optimization algorithms that incorporate inequality constraints. One advantage of "constrOptim" is that it returns the Hessian matrix of the log-likelihood (the matrix of second-order partial derivatives), facilitating the straightforward estimation of standard errors (*SEs*) as the square roots of the diagonal elements of the inverse of the observed information matrix (the negative of the Hessian matrix).

6. Model comparison on real datasets

In this section, we compare the performances of the Weibull (*W*), transmuted Weibull (*TW*), CTW_{MG}, CTW_{MA}, CTW_{MR18a}, CTW_{MR18b}, CTW_{MR19} and CTW_{R23} distributions in fitting real datasets using R software (R Core Team (2025)). The datasets selected for this comparison are the four datasets that have been used for the different cubic transmutations of Weibull distribution in Granzotto et al. (2017), Rahman et al. (2019b), Aslam et al. (2020). To avoid erroneous *MLEs* values due to the possible convergence of the BFGS algorithm to local maxima which are not the *MLEs* (see, for example, Geraldo (2021), Geraldo (2025)), we tried, for each data and each model, $m = 100$ different initial parameters and retained the one leading to the maximum value of the log-likelihood.

There exist many likelihood-based model selection criteria to find out which of the compared distributions best fits a given dataset. In this work, we will use the double of the negative log-likelihood (-2ℓ), the Akaike Information Criterion *AIC* $= -2\ell + 2k$ and the Bayesian Information Criterion *BIC* $= -2\ell + k \log n$, where ℓ is the maximum log-likelihood value, k is the number of parameters to estimate, and n is the size of the dataset. For each dataset and for each criterion, the best model is the one with the smallest value. It is not always possible that the same model simultaneously achieves the smallest values across all criteria. As noted by Granzotto et al. (2017), *AIC* and *BIC*, which incorporate penalty terms that increase with the number k of parameters, tend to favour distributions with fewer parameters.

6.1. Wheaton river data

These data, originating from Choulakian and Stephens (2001), consist of 72 peak flood exceedances (in m^3/s) of the Wheaton River (Canada) from the years 1958 to 1984, rounded to the first decimal place. Tables 1 (see page 4114) and 2 (see page 4114) respectively present the estimation results and the selection criteria for the Wheaton river data. According to the -2ℓ criterion, CTW_{R23} is the best distribution while according to the *AIC* and *BIC*, CTW_{MR19} is the best. Figures 5 and 6 respectively represent the fitted *cdfs* and *pdfs* for the Wheaton river data.

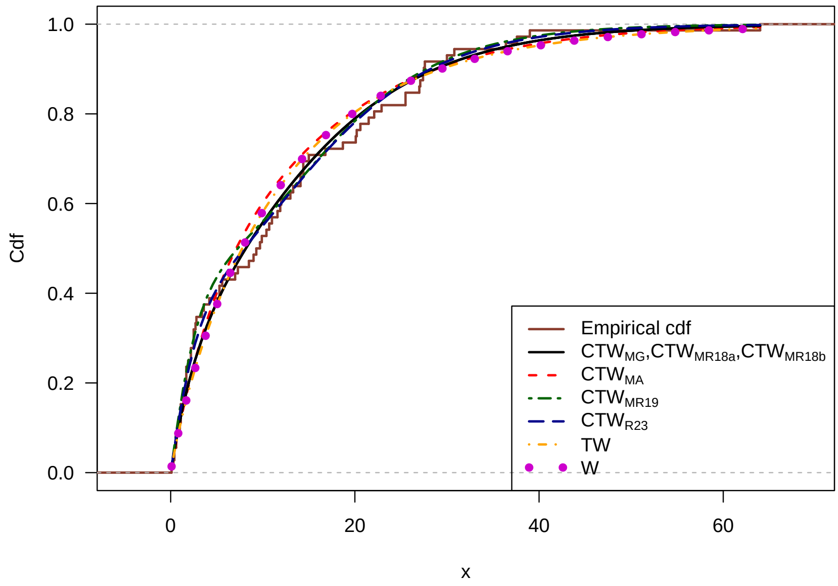


Fig. 5. Empirical and fitted cumulative distribution functions for the Wheaton river data. The fitted curves correspond to the proposed cubic transmuted Weibull models: CTW_{MG}, CTW_{MR18a}, CTW_{MR18b}, CTW_{MA}, CTW_{MR19}, CTW_{R23}, as well as the baseline Weibull (*W*) and transmuted Weibull (*TW*) distributions. This figure illustrates the comparative fit of each model to the observed data.

6.2. Carbon fibres data

These data, originating from Nichols and Padgett (2006), consist of breaking stress of carbon fibres (in GPa). Tables 3 (see page 4115) and 4 (see page 4115) respectively present the estimation results and the selection criteria for the carbon fibres data. According to the -2ℓ criterion, CTW_{MG}, CTW_{MR18a} and CTW_{MR18b} are the best distributions. Figures 7 and 8 respectively represent the fitted *cdfs* and *pdfs* for the carbon fibres data.

6.3. Remission times of bladder cancer

These data, originating from Lee and Wang (2003), consist of the remission times (in months) of 128 bladder cancer patients. Tables 5 (see page 4116) and 6 (see

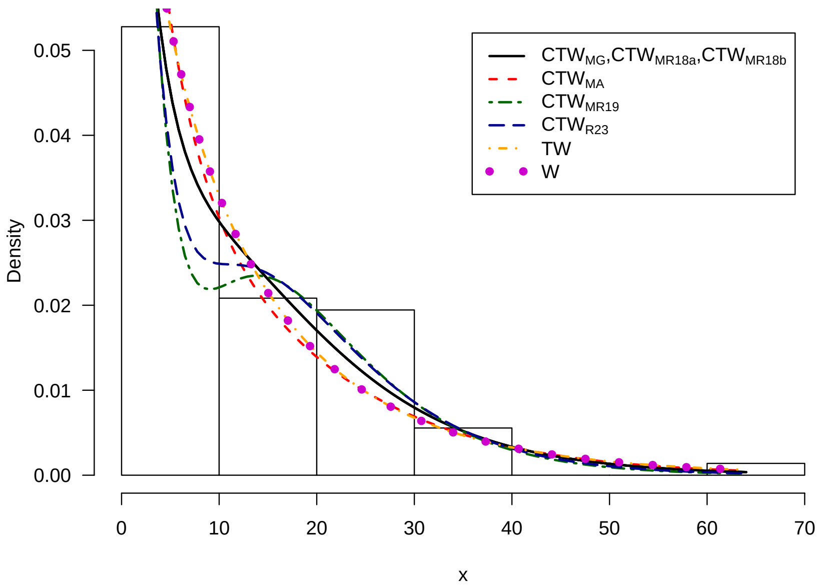


Fig. 6. Fitted probability density functions for the Wheaton river data. The fitted curves correspond to the proposed cubic transmuted Weibull models: ta. The fitted curves correspond to the proposed cubic transmuted Weibull models: CTW_{MG}, CTW_{MR18a}, CTW_{MR18b}, CTW_{MA}, CTW_{MR19}, CTW_{R23}, as well as the baseline Weibull (*W*) and transformed Weibull (*TW*) distributions. This figure highlights how well each model captures the shape of the observed data.

page 4116) respectively present the estimation results and the selection criteria for the remission times data. According to the -2ℓ criterion, CTW_{MG}, CTW_{MR18a} and CTW_{MR18b} are the best distributions while according to the *AIC* and *BIC*, CTW_{MA} is the best distribution. Figures 9 and 10 respectively represent the fitted *cdfs* and *pdfs* for the remission times data.

7. Conclusion

In this paper, we considered the *CTW* distribution. Among the six most used *CT* formulas developed in the literature over the years (see Geraldo et al. (2024) for more details), only four have been applied to the Weibull distribution so far, and the resulting *CTW* distributions have not yet been compared. We proposed a unified formulation for *CTW* distributions and derive its main statistical properties and we applied it to derive two new *CTW* distributions based on the two remaining *CT* formulas not yet utilized. The empirical comparison of the six *CTW* distributions on three real datasets suggests that all forms of *CTW* distributions are relevant for

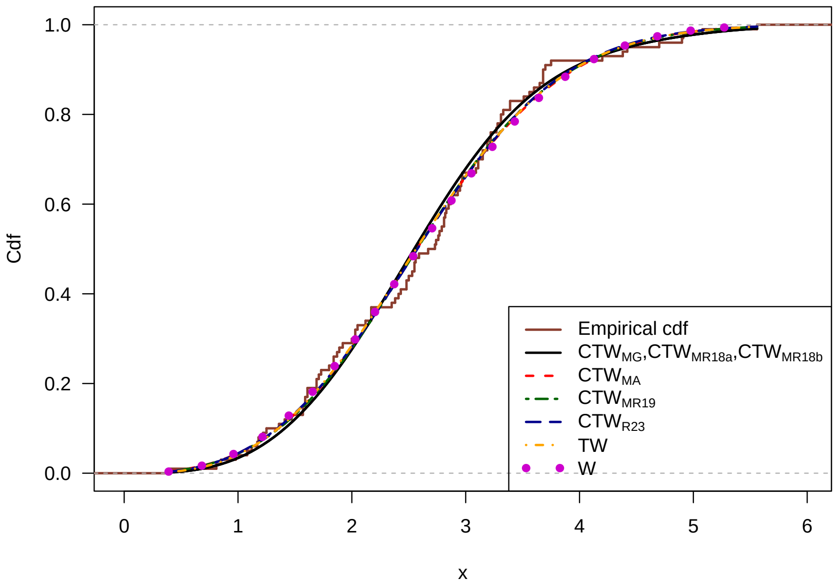


Fig. 7. Empirical and fitted cumulative distribution functions for the carbon fibres data. The fitted curves correspond to the cubic transmuted Weibull models: CTW_{MG}, CTW_{MR18a}, CTW_{MR18b}, CTW_{MA}, CTW_{MR19}, CTW_{R23}, as well as the baseline Weibull (W) and transmuted Weibull (TW) distributions. This figure illustrates the comparative fit of each model to the observed data.

data modelling. Our unified formula and its related results account for not only existing *CTW* distributions but also all potential forms of *CTW* distributions that might be developed in the future.

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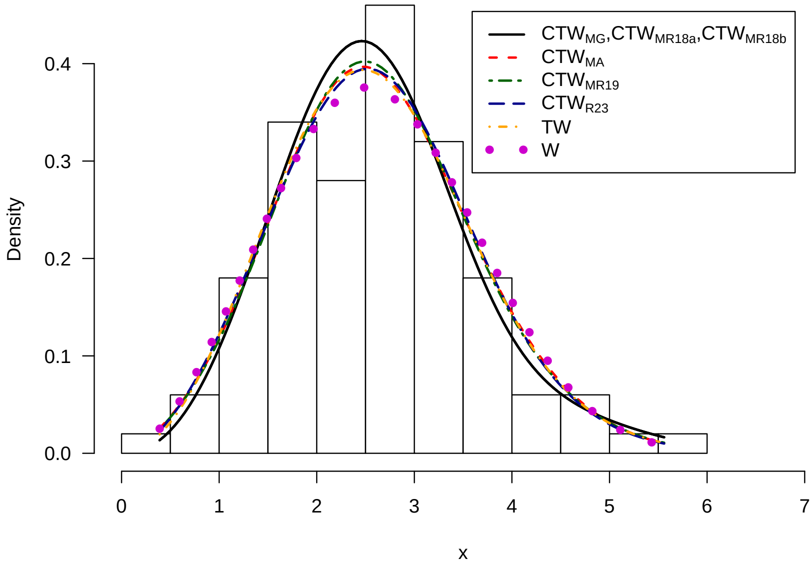

Fig. 8. Fitted probability density functions for the carbon fibres data. The fitted curves correspond to the cubic transmuted Weibull models: CTW_{MG}, CTW_{MR18a}, CTW_{MR18b}, CTW_{MA}, CTW_{MR19}, CTW_{R23},, as well as the baseline Weibull (*W*) and transmuted Weibull (*TW*) distributions. This figure highlights how well each model captures the shape of the observed data.

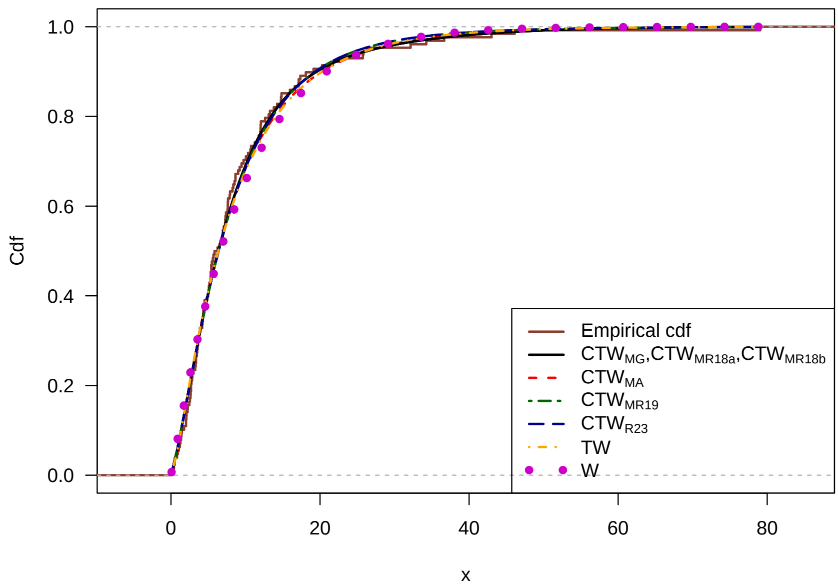

Fig. 9. Empirical and fitted cumulative distribution functions for the remission times of bladder cancer data. The fitted curves correspond to the cubic transmuted Weibull (*CTW*) models: CTW_{MG}, CTW_{MR18a}, CTW_{MR18b}, CTW_{MA}, CTW_{MR19}, CTW_{R23}, as well as the baseline Weibull (*W*) and transmuted Weibull (*TW*) distributions. This figure illustrates the comparative fit of each model to the observed survival data.

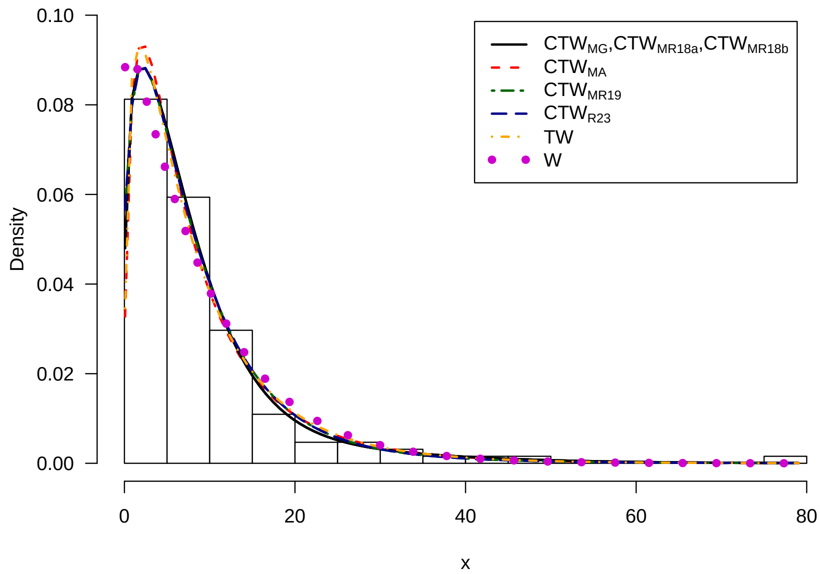

Fig. 10. Fitted probability density functions for the remission times of bladder cancer data. The fitted curves correspond to the cubic transmuted Weibull (*CTW*) models: CTW_{MG}, CTW_{MR18a}, CTW_{MR18b}, CTW_{MA}, CTW_{MR19}, CTW_{R23}, as well as the baseline Weibull (*W*) and transformed Weibull (*TW*) distributions. This figure highlights how well each model captures the shape of the observed survival data.

Table 1. Parameter estimation results for the Wheaton river data using various cubic transmuted Weibull models, baseline Weibull (*W*), and transmuted Weibull (*TW*) distributions. The table summarizes the estimated parameters and indicates model flexibility in fitting the observed data.

Distributions	*MLEs* and *SEs*			
CTW_{MG}	$\hat{\beta} = 10.544$	$\hat{\mu} = 0.996$	$\hat{\lambda}_1 = 1.381$	$\hat{\lambda}_2 = 0.000$
	(1.762)	(0.122)	(0.488)	(0.953)
CTW_{MA}	$\hat{\beta} = 12.955$	$\hat{\mu} = 1.010$	$\hat{\lambda} = 0.561$	
	(1.887)	(0.113)	(0.423)	
CTW_{MR18a}	$\hat{\beta} = 10.545$	$\hat{\mu} = 0.996$	$\hat{\lambda}_1 = 0.381$	$\hat{\lambda}_2 = -1.000$
	(1.762)	(0.122)	(0.488)	(0.954)
CTW_{MR18b}	$\hat{\beta} = 10.545$	$\hat{\mu} = 0.996$	$\hat{\lambda}_1 = -0.619$	$\hat{\lambda}_2 = 1.000$
	(1.762)	(0.122)	(0.603)	(0.951)
CTW_{MR19}	$\hat{\beta} = 10.218$	$\hat{\mu} = 1.084$	$\hat{\lambda} = -1.091$	
	(1.228)	(0.086)	(0.319)	
CTW_{R23}	$\hat{\beta} = 10.292$	$\hat{\mu} = 1.062$	$\hat{\lambda} = -1.000$	$\hat{\theta} = 1.849$
	(1.408)	(0.093)	(0.491)	(0.464)
TW	$\hat{\beta} = 11.407$	$\hat{\mu} = 0.896$	$\hat{\lambda} = -0.031$	
	(3.147)	(0.111)	(0.384)	
W	$\hat{\beta} = 11.632$	$\hat{\mu} = 0.901$		
	(1.602)	(0.086)		

Table 2. Model selection criteria for the Wheaton river data, including -2ℓ, *AIC*, and *BIC*, for various cubic transmuted Weibull (*CTW*) models, baseline Weibull (*W*), and transmuted Weibull (*TW*) distributions. According to -2ℓ, CTW_{R23} is the best-fitting distribution, while *AIC* and *BIC* favor CTW_{MR19}.

Distribution	-2ℓ	*AIC*	*BIC*
CTW_{MG}	$497.752^{(3)}$	$505.752^{(3)}$	$514.859^{(5)}$
CTW_{MA}	$501.462^{(6)}$	$507.462^{(7)}$	$514.292^{(4)}$
CTW_{MR18a}	$497.752^{(3)}$	$505.752^{(3)}$	$514.859^{(5)}$
CTW_{MR18b}	$497.752^{(3)}$	$505.752^{(3)}$	$514.859^{(5)}$
CTW_{MR19}	$495.112^{(2)}$	$501.112^{(1)}$	$507.942^{(1)}$
CTW_{R23}	$495.018^{(1)}$	$503.018^{(2)}$	$512.125^{(3)}$
TW	$502.990^{(7)}$	$508.990^{(8)}$	$515.820^{(8)}$
W	$502.998^{(8)}$	$506.998^{(6)}$	$511.551^{(2)}$

Table 3. Parameter estimation results for the carbon fibres data using various cubic transmuted Weibull (*CTW*) models, baseline Weibull (*W*), and transmuted Weibull (*TW*) distributions. The table presents the estimated parameters and illustrates model performance in capturing the data structure.

Distributions	*MLEs* and *SEs*			
CTW_{MG}	$\hat{\beta} = 3.825$	$\hat{\mu} = 3.253$	$\hat{\lambda}_1 = 2.709$	$\hat{\lambda}_2 = 0.036$
	(0.323)	(0.410)	(0.794)	(0.922)
CTW_{MA}	$\hat{\beta} = 2.813$	$\hat{\mu} = 2.497$	$\hat{\lambda} = -0.451$	
	(0.240)	(0.532)	(0.709)	
CTW_{MR18a}	$\hat{\beta} = 3.825$	$\hat{\mu} = 3.253$	$\hat{\lambda}_1 = 1.709$	$\hat{\lambda}_2 = -0.964$
	(0.323)	(0.410)	(0.794)	(0.922)
CTW_{MR18b}	$\hat{\beta} = 3.825$	$\hat{\mu} = 3.253$	$\hat{\lambda}_1 = 0.745$	$\hat{\lambda}_2 = 0.964$
	(0.323)	(0.410)	(0.237)	(0.922)
CTW_{MR19}	$\hat{\beta} = 2.969$	$\hat{\mu} = 2.554$	$\hat{\lambda} = 0.330$	
	(0.117)	(0.422)	(0.449)	
CTW_{R23}	$\hat{\beta} = 3.212$	$\hat{\mu} = 2.798$	$\hat{\lambda} = 0.575$	$\hat{\theta} = 0.693$
	(0.578)	(0.675)	(0.520)	(2.209)
TW	$\hat{\beta} = 3.413$	$\hat{\mu} = 2.993$	$\hat{\lambda} = 0.679$	
	(0.338)	(0.241)	(0.380)	
W	$\hat{\beta} = 2.944$	$\hat{\mu} = 2.793$		
	(0.111)	(0.214)		

Table 4. Model selection criteria for the carbon fibres data, including -2ℓ, *AIC*, and *BIC*, for various cubic transmuted Weibull (*CTW*) models, baseline Weibull (*W*), and transformed Weibull (*TW*) distributions. According to -2ℓ, CTW_{MG}, CTW_{MR18a}, and CTW_{MR18b} provide the best fit.

Distribution	-2ℓ	*AIC*	*BIC*
CTW_{MG}	$282.012^{(1)}$	$290.012^{(5)}$	$300.433^{(5)}$
CTW_{MA}	$282.648^{(7)}$	$288.648^{(4)}$	$296.464^{(4)}$
CTW_{MR18a}	$282.012^{(1)}$	$290.012^{(5)}$	$300.433^{(5)}$
CTW_{MR18b}	$282.012^{(1)}$	$290.012^{(5)}$	$300.433^{(5)}$
CTW_{MR19}	$282.396^{(6)}$	$288.396^{(3)}$	$296.212^{(3)}$
CTW_{R23}	$282.198^{(4)}$	$290.198^{(8)}$	$300.619^{(8)}$
TW	$282.270^{(5)}$	$288.270^{(2)}$	$296.086^{(2)}$
W	$283.058^{(8)}$	$287.058^{(1)}$	$292.268^{(1)}$

Table 5. Parameter estimation results for the remission times of bladder cancer data using various cubic transmuted Weibull (*CTW*) models, baseline Weibull (*W*), and transmuted Weibull (*TW*) distributions. The table summarizes the estimated parameters, highlighting model flexibility in fitting survival data.

Distributions	*MLEs* and *SEs*			
CTW_{MG}	$\hat{\beta} = 18.428$	$\hat{\mu} = 1.247$	$\hat{\lambda}_1 = 2.731$	$\hat{\lambda}_2 = 0.000$
	(3.139)	(0.103)	(0.537)	(0.597)
CTW_{MA}	$\hat{\beta} = 7.149$	$\hat{\mu} = 0.832$	$\hat{\lambda} = -0.953$	
	(0.746)	(0.072)	(0.165)	
CTW_{MR18a}	$\hat{\beta} = 18.428$	$\hat{\mu} = 1.247$	$\hat{\lambda}_1 = 1.731$	$\hat{\lambda}_2 = -1.000$
	(3.140)	(0.103)	(0.537)	(0.597)
CTW_{MR18b}	$\hat{\beta} = 8.564$	$\hat{\mu} = 0.828$	$\hat{\lambda}_1 = 0.503$	$\hat{\lambda}_2 = -1.433$
	(1.669)	(0.084)	(0.466)	(0.518)
CTW_{MR19}	$\hat{\beta} = 9.851$	$\hat{\mu} = 0.845$	$\hat{\lambda} = 0.757$	
	(0.926)	(0.103)	(0.242)	
CTW_{R23}	$\hat{\beta} = 9.852$	$\hat{\mu} = 0.845$	$\hat{\lambda} = 0.757$	$\hat{\theta} = 2.000$
	(1.423)	(0.135)	(0.243)	(0.582)
TW	$\hat{\beta} = 4.850$	$\hat{\mu} = 0.761$	$\hat{\lambda} = -1.000$	
	(0.706)	(0.069)	(0.119)	
W	$\hat{\beta} = 9.400$	$\hat{\mu} = 1.047$		
	(0.839)	(0.067)		

Table 6. Model selection criteria for the remission times of bladder cancer data, including -2ℓ, *AIC*, and *BIC*, for various cubic transmuted Weibull (*CTW*) models, baseline Weibull (*W*), and transformed Weibull (*TW*) distributions. According to -2ℓ, CTW_{MG}, CTW_{MR18a}, and CTW_{MR18b} are the best-fitting distributions, while *AIC* and BIC favor CTW_{MA}.

Distribution	-2ℓ	*AIC*	*BIC*
CTW_{MG}	$814.994^{(1)}$	$822.994^{(3)}$	$834.402^{(5)}$
CTW_{MA}	$815.932^{(4)}$	$821.932^{(1)}$	$830.488^{(1)}$
CTW_{MR18a}	$814.994^{(1)}$	$822.994^{(3)}$	$834.402^{(5)}$
CTW_{MR18b}	$814.994^{(1)}$	$822.994^{(3)}$	$834.402^{(5)}$
CTW_{MR19}	$816.154^{(5)}$	$822.154^{(2)}$	$830.710^{(2)}$
CTW_{R23}	$816.154^{(5)}$	$824.154^{(7)}$	$835.562^{(8)}$
TW	$817.162^{(7)}$	$823.162^{(6)}$	$831.718^{(3)}$
W	$823.878^{(8)}$	$827.878^{(8)}$	$833.582^{(4)}$

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