



Central limit theorems for associated possibly moving partial sums and application to the non-stationary invariance principles

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Received on June 06, 2025; Accepted on June 20, 2025; Published online on June 25, 2025

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Abstract. The general Central limit theorem problem deals with weak limits (in type) of sums of row-elements of array random variables. In some situations as in the invariance principle problem, the sums may include only parts of the row-elements. For strictly stationary arrays (stationary for each row), there is no change to the asymptotic results. (see page 3914 for the full abstract)

Key words: central limit theorem; Gaussian law; non-stationary associated data; Newman's approximation lemma for associated sequences; statistical applications in Actuarial Sciences

AMS 2010 Mathematics Subject Classification Objects : 60F05; 60F17; 62P05.

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Full abstract in English. The general Central limit theorem problem deals with weak limits (in type) of sums of row-elements of array random variables. In some situations as in the invariance principle problem, the sums may include only parts of the row-elements. For strictly stationary arrays (stationary for each row), there is no change to the asymptotic results. But for non-stationary data, especially for dependent data, asymptotic laws of partial sums that are moving in each row may require extra-conditions to exist. This paper deals with central limit theorems with Gaussian limits for non-stationary data. Our main focus is on dependent data, particularly on associated data. But the non-stationary independent data is also studied as a learning process. The results are applied to finite-distributional invariance principles for the types of data described above. In Moreover, results for associated sequences are interesting and innovative. Beyond their own interest, the results are expected to be applied for random sums of random variables and next in statistical modeling in many disciplines, in Actuarial sciences for example.

Résumé. (Abstract in French) Le problème général du théorème central limite traite des limites faibles (en type) des sommes par ligne de tableaux de variables aléatoires. Dans certaines situations, comme dans le problème des principes d'invariances, les sommes peuvent n'inclure qu'une partie des variables par ligne du tableau. Pour les tableaux de variables strictement stationnaires (stationnaires par ligne), les résultats asymptotiques restent inchangés. En revanche, pour les données non stationnaires, spécialement les données dépendantes, les lois asymptotiques des sommes partielles mobiles sur chaque ligne peuvent nécessiter l'existence de conditions supplémentaires. Cet article traite des théorèmes centraux limites avec des lois limites gaussiennes pour les données non stationnaires. Nous nous concentrons principalement sur les données dépendantes, en particulier les données associées. Cependant, les données indépendantes non stationnaires sont également étudiées comme un processus d'apprentissage. Les résultats sont appliqués aux principes d'invariance de distribution finie pour les types de données décrites ci-dessus. De plus, les résultats pour les données associées sont intéressants et innovants. Au-delà de leur propre intérêt, les résultats sont susceptibles d'être appliqués aux sommes aléatoires de variables aléatoires et, par la suite, à la modélisation statistique dans de nombreuses disciplines, notamment en sciences actuarielles.

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1. Introduction

Moving partial sums are closely related to invariance principles, which in turn play an important role in many areas of applications such as Finance, Actuarial Sciences, Demography, etc. As an example, consider the claims problem for an insurer, whose clients subscribe to specific products through determined policies. In vehicle insurance for instance, the policy may include that at each accident, the client makes a claim X , which depends on many factors as the severity of the crash for instance. For simplicity, we suppose that the claims are reported at discrete times nt_0 , where t_0 is a fixed period of time that may be in days, weeks or months. At each time jt_0 , the claim is a random variable X_j . So, at time nt_0 , the total claim (referred as the total loss) up to time nt_0 is given by the equation

$$S_n = \sum_{1 \leq j \leq n} X_j. \quad (1)$$

The insurer should have an accurate estimation of S_n to fix the premiums that clients should pay before the establishment of the policies, otherwise the ruin would be highly probable. We remark that the discrete time modeling of claims (1) can be extended to a continuous time one. In such a case, the number of reported claims up to time t , say $N(t)$, is a random variable and the total loss up to t is

$$S_t = \sum_{j=1}^{N(t)} X_j.$$

Now, we suppose that the insurer has a capital u at the beginning, and that the premiums can be linearized, say as ct . Then surplus process (measuring the financial balance of the insurer) at time t can be given as

$$P_t = u + ct - S_t.$$

Although the model uses continuous time, in practice, the time is discretized into multiples of a unit of time $t_0 > 0$ and the model becomes, for $t_n = nt_0$, $n \geq 0$,

$$P_{t_n} = u + ct_n - \sum_{j=1}^{N(t_n)} X_j =: u + ct_n - c_n \frac{\sum_{j=1}^{N(t_n)} X_j}{c_n}, \quad n \geq 0.$$

Finding the limiting law of the stochastic process

$$\left\{ Y_n(t) = \frac{\sum_{j=1}^{N(t_n)} X_j}{c_n}, \quad 0 \leq t \leq T, \quad n \geq 1 \right\},$$

for $T > 0$, for an appropriate sequence of normalization coefficients $(c_n)_{n \geq 1}$, to a stochastic process $\{Y(t), 0 \leq t \leq T\}$ is the essence of the invariance principle problem (or functional central limit theorem). For independent data, the most used limiting law is a Brownian motion. But, even in that case, the general solution is a Lévy process Y .

Usually, $N(t)$ is taken as a Poisson process. It is reasonable to expect that at least, the insurer should avoid incurring a ruin, say at a time $t_{n(r)}$, such that $P_{t_{n(r)}} < 0$. An approximation of the probability ruin is given by

$$P_{t_{n(r)}} \approx u + ct_{n(r)} - Y(t_{n(r)}), \quad n \geq 0, \quad (2)$$

and

$$n(r) = \inf\{n \geq 1, P_{t_n} < 0\}.$$

If Y is accurately estimated, Equation (2) may help in pre-setting c and u before contracting policies, to ensure profit and avoid ruin. To learn more on such modeling, the reader is directed to Klugman *et al.* (2004) page 252, Grandell (1991) and the references therein. For independent and square integrable data, the class of possible weak limits is exactly the class of infinitely decomposable laws and the associated invariance principle leads to Levy processes, the Brownian motion and the Poisson process being among them (see Loève (1997), Applebaum (2004), Niang *et al.* (2021), etc.).

The general problem of finding the weak law, in the random scheme as

$$S_n(t) = \sum_{j=1}^{N([nt])} X_j, \quad 0 \leq t \leq T, \quad n \geq 1$$

and, in the non-random scheme as

$$S_n(t) = \sum_{j=1}^{[nt]} X_j, \quad 0 \leq t \leq T, \quad n \geq 1,$$

(usually for $T = 1$) is the core of the invariance principle problem. This problem is hard and quite general since we do not necessarily know the dependence type between the losses X_j 's nor do we always have that $N(\circ)$ is a Poisson process. The main results and achievements in the literature are obtained for independent data and when N is a standard or a compound Poisson process.

The problem of moving partial sums arise in the important setting of functional weak limits. In what follows, we provide some background. Consider a sequence of centered random variables $(X_n)_{n \geq 1}$ defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Let $(p(n))_{n \geq 1}$ be an arbitrary sequence of positive integers. We define

$$S'_0 = 0, \quad S'_n = \sum_{k=p(n)+1}^{n+p(n)} X_k \quad \text{and} \quad s_n'^2 = \mathbb{V}ar(S'_n), \quad n \geq 1, \quad (3)$$

with, for all $k \geq 1$, $\sigma_k^2 = \mathbb{E}(X_k^2)$ and $F_k(x) = \mathbb{P}(X_k \leq x)$, $x \in \mathbb{R}$.

If $p(n) = 0$ for all $n \geq 1$, we find ourselves in studying the usual partial sums $S_n = X_1 + X_2 + \dots + X_n$ and the partial sums of variances $s_n^2 = \mathbb{V}ar(S_n)$.

In some situations, we may be concerned not with all the partial sums S_n beginning by the first r.v. X_1 but with partial sums that can start at any part of the sequence $(X_n)_{n \geq 1}$. In (3), the partial sums S'_n begin with the r.v. $X_{p(n)+1}$ and is the sum of all the n observations with index greater than $p(n)$. A more appropriate notation should be $S_{n,p(n)}$ that we denote as S'_n , given that the sequence $(p(n))_{n \geq 1}$ is already defined. For example, when dealing with invariance principles, we need to have the limit of the sequence of stochastic processes $\{Y_n(t), 0 \leq t \leq T, n \geq 1\}$ to some stochastic process $\{Y(t), 0 \leq t \leq T\}$. The state of the art (see Billingsley (1968), van der Vaart and Wellner (1996), Lo et al.(2016), etc.) expresses that, in order that weak convergence holds, we need to have the convergence in finite-distribution and that the sequence is uniformly tight (as in Billingsley (1968)) or asymptotically tight (as in van der Vaart and Wellner (1996)). For now, we focus on the weak convergence in finite distributions, i.e., of the vectors

$$(Y_n(t_j))_{1 \leq j \leq k} = \left(\frac{S_{[nt_j]} - S_{[nt_{j-1}]}}{s_n} \right)_{1 \leq j \leq k},$$

for $0 = t_0 < t_1 < \dots < t_k = T$ (usually $T = 1$), $k \geq 2$. It is clear that we have, for each j -th component:

$$Y_n(t_j) = \frac{S'_{[nt_j] - [nt_{j-1}]}}{s_n}, \text{ with } p(n) = [nt_{j-1}], 1 \leq j \leq k.$$

Here the relation between s_n and s'_n plays a major role as we will see later. We refer to such partial sums as moving partial sums, since the sequence $(p(n))_{n \geq 1}$ is arbitrary. As we will see, the handling weak laws of moving partial sums (MPS) can be a lot easier for independent *r.v.*'s. Nevertheless, we have to make sure that all steps are rigorously taken into account. *But, for dependent data, the situation requires more attention and can get very complicated. The problem is even more serious if the sequence $(p(n))_{n \geq 1}$ is random as expected in applications, especially in Actuarial Sciences and Finance.*

In this paper, we focus on associated data introduced in Subsection 3.1. We review the most important facts on associated data. Next, we refer the reader to Sangaré and Lo (2016) for a quick review. For much more details, Rao (2012) and Bulinski and Shashkin (2007) are more appropriate. Nevertheless, we will present a thorough review of some important results on independent data, in order to facilitate the passage to dependent data.

Hence, the purpose of this paper is to extend Gaussian central limits for independent data to Gaussian limits for MPS and to compare the classical conditions (Lyapounov and Lynderberg, Uniform Asymptotic Negligibility (UAN), Bounded Variance Hypothesis (BVH), Convergence Variance Hypothesis (CVH), etc.) for full sums and moving partial sums. We will then study how such results can be used in invariance principles to re-scaled Brownian motions. Since handling invariance principles for dependent data, here for associated data, requires similar asymptotic weak laws for MPS and their applications, this will lead to more general results in invariance principles for associated data in comparison with current achievements in particular in Oliveira (2012). The present analysis will open the door to studying asymptotic weak laws for MPS for other types of dependence and for other specific types limit laws, i.e., for any infinitely decomposable type limiting laws, in particular Poisson laws. Future research works will focus on generalizations of the results to random sums and their data-driven applications.

The rest of the paper is organized as follows. In Section 2, we study the MPS in the independence situation under the condition

$$\forall t \in (0, 1), s_{[nt]}^2 / s_n^2 \rightarrow a(t) \text{ as } n \rightarrow +\infty,$$

where $a(t)$ is a non-decreasing function in $t \in [0, 1]$. Next, we apply the results to finite-distributional invariance principles with weak convergence to re-scaled Brownian motions. We get moving versions for Lyapounov's theorem and the Lynderberg-Lévy-Feller's theorem. We give totally detailed proofs that are postponed to the (appendix) section 6. In Section 3, we deal with dependent random variables, here associated sequences. We make profit of the moving versions to significantly extend the central limit theorem of Sangaré H. and Lo, G.S (2018) which in turn is an extended version of Oliveira (2012). Getting weak limits of MPS's requires the following more complicated condition

$$\frac{1}{s_n^2} \mathbb{V}ar \left(\left\{ \sum_{h=[ns_1]+1}^{[ns_2]} X_h \right\} + \left\{ \sum_{h=[nt_1]+1}^{[nt_2]} X_h \right\} \right) \rightarrow \{a(s_2) - a(s_1)\} + \{a(t_2) - a(t_1)\},$$

for $0 \leq s_1 < s_2 < t_1 < t_2 \leq T$ ($T = 1$), where $a(t)$ is a non-decreasing function in $t \in [0, 1]$. Here again, we apply the results in the weak limits in the finite-distribution invariance principle for associated sequences. In that section, we used regrouped-data method as it is usual in weak laws on associated sequences. Therein, we can use both conditions on regrouped data and no-regrouped data. We need a whole section, say Section 4, to give the links between these two type of conditions. We close the paper with concluding remarks (in Section 5).

The obtained results will help in successfully addressing the general setting of random invariance principles using random numbers $N(nt)$ of data, $t \in [0, 1]$ in the innovative case of associated data, and beyond.

Let us proceed to the study for each type of dependence mentioned above.

2. Central limit theorems for independent random variables

In this section, we are going to check that Lyapounov's Theorem and Lynderberg-Levy-Feller's Theorem are unchanged in the moving frame. We exactly use the same proofs as in Loève (1997) but we follow the detailed proofs in Lo (2018). We will show how to use them in establishing general invariance principles for independent data at least for finite-distributions. Here, we adopt the following notation

$$s_n'^2 = \sum_{k=p(n)+1}^{n+p(n)} \sigma_k^2, \quad n \geq 1,$$

and

$$s_n^2 = \sum_{k=1}^n \sigma_k^2, \quad n \geq 1.$$

Here are the moving versions of the two main central limit theorems for independent data.

2.1. Moving versions

Theorem 1. (Lyapounov). Suppose that the X_k 's are independent and have finite $(2 + \delta)$ -moments for every $\delta > 0$ and

$$A'_n(\delta) = \frac{1}{s_n'^{2+\delta}} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}|X_k|^{2+\delta} \longrightarrow 0, \quad n \rightarrow +\infty. \quad (4)$$

Then, as $n \rightarrow +\infty$,

$$\frac{S'_n}{s'_n} \rightsquigarrow \mathcal{N}(0, 1).$$

Remark 1. As in the usual way, a version using arrays $\{X_{nk}, p(n) + 1 \leq k \leq n + p(n)\}$ is automatically written without any change in the proof.

Theorem 2. (Feller-Levy-Lynderberg). Suppose that the X_k 's are independent and have only finite second order moments. We have the equivalence between the assertions below:

$$\max_{p(n)+1 \leq k \leq n+p(n)} \left\{ \frac{\sigma_k}{s'_n} \right\}^2 \longrightarrow 0 \text{ and } \frac{S'_n}{s'_n} \rightsquigarrow \mathcal{N}(0, 1), \text{ when } n \rightarrow +\infty \quad (5)$$

and

$$\forall \epsilon > 0, \quad g_n(\epsilon) = \frac{1}{s_n'^2} \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \epsilon s'_n)} x^2 dF_k(x) \longrightarrow 0, \quad n \rightarrow +\infty. \quad (6)$$

As promised, the proofs are direct and require checking all lines in the mentioned proofs. They are given in the appendix. Let us focus on the applications to invariance principles.

Remark (R1). A first remark is that if we have

$$\limsup_{n \rightarrow +\infty} s_{n+p(n)}/s'_n =: \nu \in]0, +\infty[,$$

then the Lynderberg condition for the whole sequence implies that it holds for the sequence $\{X_{p(n)+j}, j \geq 1\}$. Indeed, the condition above implies that for any $\eta > 0$,

there exists n_0 such that for any $n \geq n_0$, we have $s_{n+p(n)} < s'_n(\nu + \eta)$. Hence, for $n \geq n_0$,

$$\begin{aligned} & \frac{1}{s_n'^2} \sum_{j=p(n)+1}^{n+p(n)} \int_{(|X_j| \geq \varepsilon s'_n)} |X_j|^2 d\mathbb{P} \\ & \leq \left(\frac{s_{n+p(n)}^2}{s_n'^2} \right) \frac{1}{s_{n+p(n)}^2} \sum_{j=p(n)+1}^{n+p(n)} \int_{(|X_j| \geq \varepsilon' s_{n+p(n)})} |X_j|^2 d\mathbb{P} \\ & \leq \left(\frac{s_{n+p(n)}^2}{s_n'^2} \right) \frac{1}{s_{n+p(n)}^2} \sum_{j=1}^{n+p(n)} \int_{(|X_j| \geq \varepsilon' s_{n+p(n)})} |X_j|^2 d\mathbb{P}, \end{aligned}$$

where $\varepsilon' = \varepsilon(\nu + \eta)^{-1}$. By letting $n \rightarrow +\infty$, we get the moving Lynderberg condition. The same remark applies for the Lyapounov condition.

2.2. Application to finite distributions limit invariance principles

Let us set

$$\{Y_n(t), t \in [0, 1]\} =: \left\{ \frac{S_{[nt]}}{s_n} 1_{(nt \geq 1)}, t \in [0, 1] \right\}.$$

The invariance principle investigates whether such a sequence of stochastic processes converges to a tight stochastic process, mainly to a re-scaled Brownian motion. Here, a moving version of the central limit theorem is useful. It may be not very hard to proceed for independent data. But the way we use will serve as a basis for more complex dependent data as we will see in the second part of the paper.

The limit of $s_{[nt]}/s_n$ plays an important role here. In the iid case with $\mathbb{E}X_1^2 = \sigma^2$, we have $s_n^2 = \text{Var}(S_n) = n\sigma^2$, which leads to the fact that

$$(H0) \quad \forall t \in [0, 1], \frac{s_{[nt]}^2}{s_n^2} \longrightarrow t, \quad \text{as } n \longrightarrow +\infty.$$

In the general case, we do not have such a simple relation. We have to set assumptions, for example

$$(H1) \forall t \in [0, 1], \frac{s_{[nt]}^2}{s_n^2} \longrightarrow a(t), \quad \text{as } n \longrightarrow +\infty,$$

where $a(t)$ is a non-decreasing function of $t \in (0, 1)$. From this, we may use the moving version of the Levy-Feller-Lynderberg's theorem to have the following result.

Theorem 3. Suppose that the X_j 's are independent, centered and square integrable. Let us suppose that the Lynderberg condition holds. Then the sequence of stochastic processes $\{Y_n(t), 0 \leq t \leq 1\}$ weakly converges in finite distributions to

$$\{W(a(t)), 0 \leq t \leq 1\},$$

where $\{W(t), 0 \leq t \leq 1\}$ is a Wiener stochastic process.

Proof. Let us set $0 = t_0 < t_1 < t_2 < \dots < t_{k+1} = 1$. We put :

$$\begin{cases} Z_n(t_1) = Y_n(t_1) - Y_n(t_0) = \frac{1}{s_n} \sum_{[nt_0] < h \leq [nt_1]} X_h \\ \vdots \\ Z_n(t_j) = Y_n(t_j) - Y_n(t_{j-1}) = \frac{1}{s_n} \sum_{[nt_{j-1}] < h \leq [nt_j]} X_h \\ \vdots \\ Z_n(t_k) = Y_n(t_k) - Y_n(t_{k-1}) = \frac{1}{s_n} \sum_{[nt_{k-1}] < h \leq [nt_k]} X_h \end{cases}.$$

For each fixed $j \in \{1, \dots, k\}$, we have

$$Z_n(t_j) = \frac{1}{s_n} \sum_{h=[nt_{j-1}]+1}^{[nt_j]} X_h = \frac{\sqrt{s_{[nt_j]}^2 - s_{[nt_{j-1}]}^2}}{\sqrt{s_n^2}} \times \frac{1}{\sqrt{s_{[nt_j]}^2 - s_{[nt_{j-1}]}^2}} \sum_{h=[nt_{j-1}]+1}^{[nt_j]} X_h.$$

Since the Lynderberg condition holds for the whole sequence, it holds for each sequence $\{X_{[nt_{j-1}]+h}, h \geq 1\}$, $j \in \{1, \dots, k\}$ [See Remark (R1), page 3920]. So, we may apply the moving Levy-Lynderberg-Feller's (CLT) for each $j \in \{1, \dots, k\}$ with $p(n) = [nt_{j-1}]$ to get that

$$\frac{1}{\sqrt{s_{[nt_j]}^2 - s_{[nt_{j-1}]}^2}} \sum_{h=[nt_{j-1}]+1}^{[nt_j]} X_h$$

weakly converges to the standard normal law $\mathcal{N}(0, 1)$ and by (H1)

$$\frac{s_{[nt_j]}^2}{s_n^2} \longrightarrow a(t_j) \text{ as } n \longrightarrow +\infty.$$

So, for each $j \in \{1, \dots, k\}$, by Slutsky's lemma, we have

$$Z_n(t_j) \rightsquigarrow \mathcal{N}(0, a(t_j) - a(t_{j-1})), \text{ as } n \longrightarrow +\infty,$$

since

$$\frac{s_{[nt_j]}^2 - s_{[nt_{j-1}]}^2}{s_n^2} \rightarrow a(t_j) - a(t_{j-1})$$

for each $j \in \{1, \dots, k\}$. Now, for any $u = (u_1, \dots, u_k) \in \mathbb{R}^k$,

$$\mathbb{E} \left(\exp i \left(\sum_{1 \leq j \leq k} Z_n(t_j) u_j \right) \right) = \prod_{1 \leq j \leq k} \mathbb{E}(\exp(i Z_n(t_j) u_j)) \rightarrow \prod_{1 \leq j \leq k} e^{-\frac{1}{2} u_j^2 (a(t_j) - a(t_{j-1}))}.$$

So, the vector $Z_n = (Z_n(t_j), 1 \leq j \leq k)^t$ weakly converges to a random Gaussian vector Z , with independent components. For each $1 \leq j \leq k$ the j^{th} component variance of Z is: $a(t_j) - a(t_{j-1})$.

By the continuous theorem mapping theorem, $Y_n = AZ_n$ with

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \dots & 0 \\ 1 & \dots & 1 & 0 \\ 1 & \dots & \dots & 1 \end{pmatrix}$$

weakly converges to $V = AZ$ and its components verify

$$V_j = Z_1 + \dots + Z_j.$$

Next we see that

$$Z_j = V_j - V_{j-1}.$$

Hence, we conclude that the finite margins of $Y_n(t)$ weakly converge to those of $W(a(t))$, where W is a Wiener process, since

$$\mathbb{E}(V_j^2) = a(t_j) \quad \text{and} \quad \mathbb{E}(V_j V_h) = a(t_j) \wedge a(t_h),$$

for all j and h both in $\{1, \dots, k\}$. ■

3. central limit theorems for associated data

3.1. Easy introduction to Associated random variables

In fear of rendering this paper heavier, we refer the reader to [Sangaré and Lo \(2016\)](#) for a quick introduction on associated sequence of random variables, and give the important Newman's inequality as follows. Readers are directed to [Rao \(2012\)](#) and [Bulinski and Shashkin \(2007\)](#) for a more detailed introduction to associations. The above mentioned inequality is used below several times.

Lemma 1 (Newman and Wright (1981) Theorem, see Newman and Wright (1981)).

Let $X_1, X_2, \dots, X_n$ be associated, then we have for all $t = (t_1, \dots, t_n) \in \mathbb{R}^n$,

$$\left| \psi_{(X_1, X_2, \dots, X_n)}(t) - \prod_{j=1}^n \psi_{X_j}(t_j) \right| \leq \frac{1}{2} \sum_{1 \leq j \neq h \leq n} |t_j t_h| \text{Cov}(X_j, X_h). \quad (7)$$

Let us begin with the approximation result.

3.2. Moving version of the central limit theorem for associated data

As in [Sangaré H. and Lo, G.S \(2018\)](#), we suppose that we have sequences of integer numbers $\ell(n)$, $m(n)$, $r(n)$ such that $n = m(n)\ell(n) + r(n)$, with $0 \leq r(n) < \ell(n)$, $0 \leq m(n) \rightarrow +\infty$ and

$$\ell(n)/n \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (L)$$

We want to stress that the integers $m = m(n)$, $\ell = \ell(n)$ and $r = r(n)$ are functions of $n \geq 1$ throughout the text even though we may and do drop the label n in many situations for simplicity's sake. Let us suppose that $(p(n))_{n \geq 1}$ is an arbitrary sequence. The moving versions corresponding to the hypotheses in [Sangaré H. and Lo, G.S \(2018\)](#) and alike papers as [Oliveira \(2012\)](#) are as follows.

We denote

$$S'_0 = 0, \quad S'_n = \sum_{h=p(n)+1}^{n+p(n)} X_h \quad \text{and} \quad s_n'^2 = \text{Var}(S'_n) \quad \text{for } n \geq 1.$$

The hypotheses we will be using are the following.

$$\frac{\ell(n)}{s_n'^2} \rightarrow 0 \text{ as } n \rightarrow +\infty; \quad (H0)$$

$$\frac{\ell(n)}{s_n'^2} \sum_{j=1}^{m(n)} \mathbb{V}ar \left(\frac{S'_{j\ell(n)} - S'_{(j-1)\ell(n)}}{\sqrt{\ell(n)}} \right) \rightarrow 1 \text{ as } n \rightarrow +\infty; \quad (\text{Ha})$$

$$\frac{1}{s_n'^2} \mathbb{V}ar \left(\sum_{j=p(n)+m(n)\ell(n)+1}^{p(n)+n} X_j \right) \rightarrow 0 \text{ as } n \rightarrow +\infty; \quad (\text{Hab})$$

$$\sup_{1 \leq j \leq m(n)} \frac{\ell(n)}{s_n'^2} \mathbb{V}ar \left(\frac{S'_{j\ell(n)} - S'_{(j-1)\ell(n)}}{\sqrt{\ell(n)}} \right) = C_1(n) \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (\text{Hb})$$

In the sequel, it may be handy to use the notation

$$Y_{j,\ell} = \frac{S'_{j\ell(n)} - S'_{(j-1)\ell(n)}}{\sqrt{\ell(n)}}, 1 \leq j \leq m = m(n). \quad (8)$$

Let us prove the moving version of Formula (4.12) in Sangaré H. and Lo, G.S (2018).

Proposition 1. Under Hypotheses (L), (H0), (Ha), (Hab) and (Hb), we have for any $t \in \mathbb{R}$, as $n \rightarrow +\infty$,

$$\left| \Psi_{\frac{s'_n}{s_n}}(t) - \prod_{j=1}^m \Psi_{Y_{j,\ell}} \left(\frac{\sqrt{\ell}}{s'_n} t \right) \right| \rightarrow 0. \quad (9)$$

Proof. We follow the one in Sangaré H. and Lo, G.S (2018) which we appropriately adapt. Let us denote

$$\Psi_{\frac{s'_n}{s_n}}(t) = \mathbb{E} \left(e^{itS'_n/s'_n} \right), t \in \mathbb{R}.$$

We have for $t \in \mathbb{R}$,

$$\left| \Psi_{\frac{s'_n}{s_n}}(t) - \Psi_{\frac{s'_{m\ell}}{s'_n}}(t) \right| = \left| \mathbb{E}(e^{itS'_n/s'_n}) - \mathbb{E}(e^{itS'_{m\ell}/s'_n}) \right| \quad (10)$$

$$= \left| \mathbb{E} \left[e^{itS'_{m\ell}/s'_n} \left(e^{it[(S'_n/s'_n) - (S'_{m\ell}/s'_n)]} - 1 \right) \right] \right| \quad (11)$$

$$\leq \mathbb{E} \left| e^{it \left(\frac{s'_n}{s'_n} - \frac{s'_{m\ell}}{s'_n} \right)} - 1 \right|. \quad (12)$$

But for any $x \in \mathbb{R}$,

$$|e^{ix} - 1| = |(\cos x - 1) + i \sin x| = \left| 2 \sin \frac{x}{2} \right| \leq |x|.$$

Thus the second member of (12) is, by the Cauchy-Schwarz's inequality, bounded by

$$|t| \mathbb{E} \left| \frac{S'_n}{s'_n} - \frac{S'_{m\ell}}{s'_n} \right| \leq |t| \mathbb{V}ar \left(\frac{S'_n}{s'_n} - \frac{S'_{m\ell}}{s'_n} \right)^{\frac{1}{2}}$$

and

$$\delta_{m,\ell} = \mathbb{V}ar \left(\frac{S'_n}{s'_n} - \frac{S'_{m\ell}}{s'_n} \right) = \frac{1}{s_n'^2} \mathbb{V}ar (S'_n - S'_{m\ell}),$$

which tends to zero as $n \rightarrow +\infty$ by (Hab) since

$$\begin{aligned} \delta_{m,\ell} &= \frac{1}{s_n'^2} \mathbb{V}ar (S'_n - S'_{m\ell}) \\ &= \frac{1}{s_n'^2} \mathbb{V}ar \left(\sum_{j=p(n)+m(n)\ell(n)+1}^{p(n)+n} X_j \right) \\ &\rightarrow 0. \end{aligned}$$

This proves that

$$\left| \Psi_{\frac{S'_n}{s'_n}}(t) - \Psi_{\frac{S'_{m\ell}}{s'_n}}(t) \right| \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (13)$$

Next, recall that $Y_{j,\ell} = (S'_{j\ell} - S'_{(j-1)\ell})/\sqrt{\ell}$, for $1 \leq j \leq m$. Observe that

$$\frac{S'_{m\ell}}{s'_n} = \frac{\sqrt{\ell}}{s'_n} \sum_{j=1}^m Y_{j,\ell}.$$

According to the Newman's inequality (see Lemma 1), we have

$$\left| \Psi_{\frac{S'_{m\ell}}{s'_n}}(t) - \prod_{j=1}^m \Psi_{Y_{j,\ell}} \left(\frac{\sqrt{\ell}}{s'_n} t \right) \right| \leq \frac{\ell t^2}{2s_n'^2} \sum_{1 \leq j \neq k \leq m} \text{Cov}(Y_{j,\ell}, Y_{k,\ell}).$$

But,

$$\begin{aligned} \frac{\ell t^2}{2s_n'^2} \sum_{1 \leq j \neq k \leq m} \text{Cov}(Y_{j,\ell}, Y_{k,\ell}) &= \frac{\ell t^2}{2s_n'^2} \mathbb{V}ar \left(\sum_{j=1}^m Y_{j,\ell} \right) - \frac{\ell t^2}{2s_n'^2} \sum_{j=1}^m \mathbb{V}ar(Y_{j,\ell}) \\ &= \frac{t^2}{2} \left[\mathbb{V}ar \left(\frac{\sqrt{\ell}}{s'_n} \sum_{j=1}^m Y_{j,\ell} \right) - \frac{\ell}{s_n'^2} \sum_{j=1}^m \mathbb{V}ar(Y_{j,\ell}) \right] \\ &= \frac{t^2}{2} \left[\mathbb{V}ar \left(\frac{1}{s'_n} S'_{m\ell} \right) - \frac{\ell}{s_n'^2} \sum_{j=1}^m \mathbb{V}ar \left(\frac{S_{j\ell} - S_{(j-1)\ell}}{\sqrt{\ell}} \right) \right]. \quad (L3) \end{aligned}$$

From (L3), we use

$$\frac{1}{s'_n} S'_n = \frac{1}{s'_n} S'_{m\ell} + \frac{1}{s'_n} \sum_{j=p(n)+m\ell+1}^{n+p(n)} X_j,$$

and take the associativity into account to get

$$1 = \mathbb{V}ar \left(\frac{1}{s'_n} S'_n \right) \geq \mathbb{V}ar \left(\frac{1}{s'_n} S'_{m\ell} \right) + \mathbb{V}ar \left(\frac{1}{s'_n} \sum_{j=p(n)+m\ell+1}^{n+p(n)} X_j \right).$$

This leads to

$$\begin{aligned} & \frac{\ell t^2}{2s_n'^2} \sum_{1 \leq j \neq k \leq m} \text{Cov}(Y_{j,\ell}, Y_{k,\ell}) \\ & \leq \frac{t^2}{2} \left[1 - \frac{\ell}{s_n'^2} \sum_{j=1}^m \mathbb{V}ar \left(\frac{S'_{j\ell} - S'_{\ell(j-1)}}{\sqrt{\ell}} \right) \right] \\ & \quad - \frac{t^2}{2s_n'^2} \mathbb{V}ar \left(\sum_{j=1}^r X_{p(n)+m\ell+j} \right). \end{aligned}$$

Thus, by (Ha) and (Hab), we get, as $n \rightarrow +\infty$ our conclusion, i.e.,

$$\left| \Psi_{\frac{S'_{m\ell}}{s'_n}}(t) - \prod_{j=1}^m \Psi_{Y_{j,\ell}} \left(\frac{\sqrt{\ell}}{s'_n} t \right) \right| \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (14)$$

Let us draw a first important application of our results regarding the invariance principle.

3.3. Gaussian Central limit theorem

Proposition 1 says that under the hypotheses (L), (H0), (Ha), (Hab) and (Hb), S'_n/s'_n has the same weak limit law than

$$\frac{1}{s'_n} \sum_{j=1}^{m(n)} T_{j,n},$$

where, for each $n \geq 1$, $(T_{1,n}, \dots, T_{m(n),n})$ has independent components and for each $1 \leq j \leq m(n)$

$$T_{j,n} =_d S'_{j\ell(n)} - S'_{(j-1)\ell(n)},$$

where the random variables in the left-hand of the equality are already defined in Formula (8) and $=_d$ stands for the equality in distribution.

A - Lyapounov central limit theorem. As a consequence a Lyapounov condition for independent data is enough to ensure the Gaussian central limit theorem. Let us set for $\delta > 0$,

$$\frac{1}{s_n^{2+\delta}} \sum_{j=1}^m \mathbb{E} \left| S'_{j\ell(n)} - S'_{(j-1)\ell(n)} \right|^{2+\delta} = C_2(n) \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (\text{Hc})$$

By the C_r -inequality ($|a + b|^r \leq 2^{r-1}(|a|^r + |b|^r)$ for real-valued numbers a, b and $r \geq 1$), the $S'_{j\ell(n)} - S'_{(j-1)\ell(n)}$'s have finite $(2 + \delta)$ -moments, $\delta > 0$ if the X_j 's do.

So, we have

Theorem 4. Let $\delta > 0$. If the X_j 's have finite $(2 + \delta)$ -moments and (Hc) holds on top of the hypotheses (L), (HO), (Ha), (Hab) and (Hb), then

$$S'_n / s'_n \rightsquigarrow \mathcal{N}(0, 1).$$

Proof Since the study transformed into that of some in independent and centered data, the Lyapounov condition is enough to get the conclusion, but the involved variance is $s_{m\ell}^{\prime 2}$ which is equivalent to $s_n^{\prime 2}$ by Hypotheses (Ha).

B - A Feller-Levy-Lynderberg central limit theorem.

With the equivalence $s_{m\ell}^{\prime 2} \sim s_n^{\prime 2}$ by Hypothesis (Ha), the Lynderberg condition on the $T_{j,n}$'s is equivalent to: $\forall \epsilon > 0$,

$$\frac{1}{s_n^{\prime 2}} \sum_{j=1}^{m(n)} \int_{(|S'_{j\ell(n)} - S'_{(j-1)\ell(n)}| \geq \epsilon \tau'_n)} |S'_{j\ell(n)} - S'_{(j-1)\ell(n)}|^2 d\mathbb{P} \longrightarrow 0, \quad (15)$$

where

$$\tau_{j,n}^{\prime 2} = \mathbb{V}ar(S'_{j\ell(n)} - S'_{(j-1)\ell(n)}), \quad 1 \leq j \leq m(n)$$

and

$$\tau_n'^2 = \sum_{j=1}^{m(n)} \tau_{j,n}'^2.$$

We denote

$$B_n = \max\{\tau_{j,n}'/\tau_n', 1 \leq j \leq m(n)\}.$$

The Feller-Levy-Lynderberg (FLL) theorem is stated as follows.

Theorem 5. *We suppose that the X_j 's have finite second order moments and that the hypotheses (L), (H0), (Ha), (Hab) and (Hb) hold. Then*

$$S_n'/s_n' \rightsquigarrow \mathcal{N}(0,1) \text{ and } B_n \rightsquigarrow 0$$

if and only if the Lynderberg condition (15) holds.

Here, we directly used the Lyapounov and the FLL conditions ((Hc) and 15) for on the regrouped data (the $T_{j,n}$'s). It may be more convenient to give sufficient conditions of the non-regrouped data for they hols. When we proceed to complete comparison between the situation for regrouped data and non-regrouped data as we will do it in Section , we arrive at this final version.

Theorem 6. *Suppose that the random variables $X_j, j \geq 1$, are centered and square integrable and associated and that Hypotheses (L), (H0), (Ha), (Hab) and (Hb) are satisfied. Then we have following results:*

(1) *If the X_j 's have $(2 + \delta)$ -moments for some $\delta > 0$ and the Lyapounov-type condition*

$$\frac{\ell(n)^{1+\delta}}{s_n^{1+(2+\delta)}} \sum_{p(n)+1 \leq j \leq p(n)+n} \mathbb{E}|X_j|^{2+\delta} \rightarrow 0 \text{ as } n \rightarrow +\infty, \quad (16)$$

then

$$S_n'/s_n' \rightsquigarrow \mathcal{N}(0,1) \text{ as } n \rightarrow +\infty.$$

(2) *Suppose that the following uniform negligibility of the variances*

$$\frac{\ell(n)^2}{\epsilon^2 s_n'^2} \max_{p(n)+1 \leq j \leq p(n)+n} \mathbb{E}X_j^2 \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

hold. Then we have

$$S'_n/s'_n \rightsquigarrow \mathcal{N}(0, 1) \text{ as } n \rightarrow +\infty.$$

if and only if the following Lynderberg-type condition holds:

$$L'_n\left(\frac{\varepsilon}{2\ell(n)}\right) \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (17)$$

where, for $\varepsilon > 0$ and $n \geq 1$,

$$L'_n(\varepsilon) = \frac{\ell(n)^2}{s_n'^2} \sum_{p(n)+1 \leq j \leq p(n)+n} \int_{(|X_j| > \varepsilon s'_n)} X_j^2 d\mathbb{P}.$$

As said earlier, we will give a complete justification of that theorem in Section 4.

Now, let us see the applications of the results to invariance principles.

3.4. Invariance principles of associated data

We already defined the sequence of stochastic processes

$$\{Y_n(t), t \in [0, 1]\} =: \left\{ \frac{S_{[nt]}}{s_n} 1_{(nt \geq 1)}, t \in [0, 1] \right\}$$

in page 3921 and we wish to find the weak law. Above, in the independent case, we used the hypothesis (H1) which controls how the variance $s_{[nt]}$ grows with respect to s_n . Such a condition still works if the data are stationary. Otherwise, we may need the following more elaborated assumption.

(H1-NSA). There exists a measurable function $a(t)$ of $t \in [0, 1]$ such that for $0 < s_1 < s_2 \leq t_1 < t_2 \leq 1$, as $n \rightarrow +\infty$,

$$\frac{1}{s_n^2} \mathbb{V}ar \left(\left\{ \sum_{h=[ns_1]+1}^{[ns_2]} X_h \right\} + \left\{ \sum_{h=[nt_1]+1}^{[nt_2]} X_h \right\} \right) \rightarrow \{a(s_2) - a(s_1)\} + \{a(t_2) - a(t_1)\}.$$

From this, we may use the moving version of the Levy-Feller-Lynderberg's theorem or Lyapounov theorem, we have the following result.

Theorem 7. *Let us suppose that the X_j 's are associated and (H1-NSA) holds. Let us suppose that the hypotheses (L), (H0), (Ha), (Hab) and (Hb) hold on top of the moving Lynderberg condition. Then the sequence of stochastic processes $\{Y_n(t), 0 \leq t \leq 1\}$ weakly converges in finite distributions to*

$$\{W(a(t)), 0 \leq t \leq 1\},$$

where $\{W(t), 0 \leq t \leq 1\}$ is a Wiener stochastic process.

Proof. Let us set $0 = t_0 < t_1 < t_2 < \dots < t_{k+1} = 1$. We put :

$$\begin{cases} Z_{1,n} = Y_n(t_1) - Y_n(t_0) = \frac{1}{s_n} \sum_{[nt_0] < h \leq [nt_1]} X_h \\ \vdots \\ Z_{j,n} = Y_n(t_j) - Y_n(t_{j-1}) = \frac{1}{s_n} \sum_{[nt_{j-1}] < h \leq [nt_j]} X_h \\ \vdots \\ Z_{k,n} = Y_n(t_k) - Y_n(t_{k-1}) = \frac{1}{s_n} \sum_{[nt_{k-1}] < h \leq [nt_k]} X_h \end{cases}$$

For each fixed $j \in \{1, \dots, k\}$, $s_n Z_{j,n}$ is a moving partial sum with $p(j, n) = [nt_{j-1}]$, $\Delta(j, n) = [nt_j] - [nt_{j-1}]$ and

$$s_n Z_{j,n} = \sum_{h=p(j,n)+1}^{p(j,n)+\Delta(j,n)} X_h.$$

Put, for each $j \in \{1, \dots, k\}$,

$$s_{j,n}'^2 = \mathbb{V}ar \left(\sum_{h=p(j,n)+1}^{p(j,n)+\Delta(j,n)} X_h \right).$$

We have the central limit theorem for $s_n Z_{j,n}$, i.e.,

$$\frac{s_n Z_{j,n}}{s_{j,n}'} \rightsquigarrow \mathcal{N}(0, 1),$$

and by Slutsky's lemma, we have

$$Z_{j,n} \rightsquigarrow \mathcal{N}(0, \Delta a(t_j)),$$

where we denote $\Delta a(t_j) = a(t_j) - a(t_{j-1})$, $j \in \{1, \dots, k\}$. Next, let us apply Wold Criterion (see Lo (2018), Chapter 1) and consider

$$Z_n = u_1 Z_{1,n} + \dots + u_k Z_{k,n}.$$

For $t \in \mathbb{R}$, we have

$$\begin{aligned} & \left| \psi_{Z_n}(t) - \prod_{j=1}^k \exp\left(-\frac{1}{2}\Delta a(t_j)t^2 u_j^2\right) \right| \\ & \leq \left| \psi_{Z_n}(t) - \prod_{j=1}^k \psi_{Z_{j,n}}(tu_j) \right| \\ & + \left| \prod_{j=1}^k \psi_{Z_{j,n}}(tu_j) - \prod_{j=1}^k \exp\left(-\frac{1}{2}\Delta a(t_j)t^2 u_j^2\right) \right| \\ & =: R_n(1) + R_n(2). \end{aligned}$$

We already proved that $R_n(2) \rightarrow 0$. It remains to prove that $R_n(1)$ does. But, by Newman's inequality in Lemma 1, we have

$$0 \leq R_n(1) \leq \frac{t^2}{4} \sum_{1 \leq j_1 < j_2 \leq k} |u_{j_1} u_{j_2}| \mathbb{C}ov(Z_{j_1,n}, Z_{j_2,n}).$$

Now for any $1 \leq j_1 < j_2 \leq k$,

$$2\mathbb{C}ov(Z_{j_1,n}, Z_{j_2,n}) = R_n(1, 1) - R_n(1, 2),$$

with

$$R_n(1, 1) = \frac{1}{s_n^2} \left(\mathbb{V}ar \left(\left\{ \sum_{h=[nt_{j_1-1}]+1}^{[nt_{j_1}]} X_h \right\} + \left\{ \sum_{h=[nt_{j_2-1}]+1}^{[nt_{j_2}]} X_h \right\} \right) \right)$$

and

$$R_n(1, 2) = \frac{1}{s_n^2} \mathbb{V}ar \left(\sum_{h=[nt_{j_1-1}]+1}^{[nt_{j_1}]} X_h \right) + \frac{1}{s_n^2} \mathbb{V}ar \left(\sum_{h=[nt_{j_2-1}]+1}^{[nt_{j_2}]} X_h \right),$$

which both converge to $\Delta a(t_{j_1}) + \Delta a(t_{j_2})$ and hence $2\mathbb{C}ov(Z_{j_1,n}, Z_{j_2,n}) \rightarrow 0$. From there, the conclusion is the same as in the independence case since we move from the limit to the increments to the sequence $(Y_n(t_1), \dots, Y_n(t_k))$ in the same way. $\square$

4. Weak limits of moving partial sums using condition on full data

Theorems 4 and 5 used Lyapounov and *FLL*-type conditions on the regrouped date. Using Proposition 1, we pointed out that under Hypotheses (H0), (Ha), (Hab) and (Hb), the weak limit law of $Y_n = S'_n/s'_n$ behaves exactly as that of

$$T''_n = \sum_{j=1}^{m(n)} T_{j,n}/\tau'_n,$$

where $T_{j,n} = S'_{j\ell(n)} - S'_{(j-1)\ell(n)}$, $1 \leq j \leq m(n)$, and the $T_{j,n}$ are independent, with

$$\tau_n'^2 = \sum_{j=1}^{m(n)} \tau_{j,n}'^2, \quad \tau_{j,n}'^2 = \text{Var}(T_{j,n}), \quad 1 \leq j \leq m(n).$$

From there, the theory on weak limits of independent case applies. In studying T''_n we have the Bounded Variance Hypothesis (BVH) is satisfied with

$$\sup_{n \geq 1} \sum_{1 \leq j \leq m(n)} \text{Var}(T_{j,n}/\tau'_n) \leq c,$$

for $c = 1$. But, because of Hypothesis (Ha), we may change the sequence we study to

$$T''_n = \sum_{j=1}^{m(n)} T_{j,n}/s'_n$$

and the (BVH) still holds for some $c > 0$. Moreover, the Uniformly Asymptotically Negligibility (UAN) condition in sums of independent random variables theory :

$$U''(n) = \max_{1 \leq j \leq m(n)} \mathbb{P}(|T_{j,n}| \geq \varepsilon \tau'_n) \rightarrow 0, \quad (18)$$

is controlled as follows:

$$\forall \varepsilon > 0, \quad U''(n) \leq B''_n = \frac{\varepsilon^{-2}}{\tau_n'^2} \max_{1 \leq j \leq m(n)} \tau_{j,n}'^2 \rightarrow 0. \quad (19)$$

Now, in the first place, let us see the links of usual conditions related to theory of sums of independent random variables when they are expressed on the $T_{j,n}$'s and on the X_j 's.

A - Convergence conditions on the regrouped and simple data.

(A1) The UAN condition. That condition is expressed on the regrouped data $T_{j,n}$'s. It is controlled with B_n'' . This itself is bounded through the simple data $\{X_{p(n)+j}, 1 \leq j \leq n\}$ as follows. By using the convexity of $\mathbb{R}_+ \ni x \rightarrow x^2$, we have

$$\begin{aligned} B_n'' &= \frac{1}{\epsilon^2 \tau_n'^2} \max_{1 \leq j \leq m(n)} \mathbb{E} \left(\sum_{h=1}^{\ell(n)} X_{p(n)+(j-1)\ell(n)+h} \right)^2 \\ &\leq \frac{\ell(n)}{\epsilon^2 \tau_n'^2} \max_{1 \leq j \leq m(n)} \sum_{h=1}^{\ell(n)} \mathbb{E} X_{p(n)+(j-1)\ell(n)+h}^2 \\ &\leq \frac{\ell(n)^2}{\epsilon^2 \tau_n'^2} \max_{p(n)+1 \leq j \leq p(n)+n} \mathbb{E} X_j^2. \end{aligned}$$

By denoting

$$B_n' = \frac{\ell(n)^2}{\epsilon^2 s_n'^2} \max_{p(n)+1 \leq j \leq p(n)+n} \mathbb{E} X_j^2, \quad (20)$$

we have

$$B_n'' \leq \left(\frac{s_n'}{\tau_n'} \right)^2 B_n'. \quad (21)$$

(A2) - Lyapounov condition. For $\delta > 0$, the Lyapounov condition is

$$A_n''(\delta) = \frac{1}{\tau_n'^{(2+\delta)}} \sum_{1 \leq j \leq m(n)} \mathbb{E} \left| \sum_{1 \leq h \leq \ell(n)} X_{p(n)+(j-1)\ell(n)+h} \right|^{2+\delta}. \quad (22)$$

To shorten the notation, we use the following notation below

$$X'_{n,j,h} = X_{p(n)+(j-1)\ell(n)+h}, \quad 1 \leq j \leq m(n), \quad 1 \leq h \leq \ell(n).$$

Now, by using the convexity of $\mathbb{R}_+ \ni x \rightarrow x^{2+\delta}$, we have

$$\begin{aligned} A_n''(\delta) &= \frac{1}{\tau_n'^{(2+\delta)}} \sum_{1 \leq j \leq m(n)} \mathbb{E} \left| \ell(n) \left(\sum_{1 \leq h \leq \ell(n)} \{X'_{n,j,h}/\ell(n)\} \right) \right|^{2+\delta} \\ &\leq \frac{\ell(n)^{1+\delta}}{\tau_n'^{(2+\delta)}} \sum_{1 \leq j \leq m(n)} \sum_{1 \leq h \leq \ell(n)} \mathbb{E} |X'_{n,j,h}|^{2+\delta} \\ &\leq \frac{\ell(n)^{1+\delta}}{\tau_n'^{(2+\delta)}} \sum_{p(n)+1 \leq j \leq p(n)+n} \mathbb{E} |X_j|^{2+\delta}, \end{aligned}$$

and we get

$$A_n''(\delta) \leq \left(\frac{s_n'^{(2+\delta)}}{\tau_n'^{(2+\delta)}} \right) A_n'(\delta), \quad (23)$$

where

$$A_n'(\delta) = \frac{\ell(n)^{1+\delta}}{s_n'^{(2+\delta)}} \sum_{p(n)+1 \leq j \leq p(n)+n} \mathbb{E} |X_j|^{2+\delta}. \quad (24)$$

(A3) - Lynderberg Condition. For $\varepsilon > 0$, the Lynderberg Condition for regrouped data is

$$L_n''(\varepsilon) = \frac{1}{\tau_n'^2} \sum_{j=1}^{m(n)} \int_{(|T_{j,n}| \geq \varepsilon \tau_n')} |T_{j,n}|^2 d\mathbb{P}. \quad (25)$$

Let us set, $n \geq 1$, $1 \leq j \leq m(n)$, $1 \leq h \leq \ell(n)$,

$$A_{n,j,h}' = (|X_{n,j,h}'| \geq \varepsilon \tau_n' / \ell(n))$$

and

$$A_{n,j}' = \bigcup_{h=1}^{\ell(n)} A_{n,j,h}'.$$

We have

$$L_n''(\varepsilon) \leq \frac{1}{\tau_n'^2} \sum_{j=1}^{m(n)} \int_{A_{n,j}'} |T_{j,n}|^2 d\mathbb{P}.$$

Let $M_n = \max_{1 \leq s \leq \ell(n)} |X_{n,j,s}'|$. Since we have

$$\bigcup_{1 \leq r \leq \ell(n)} \left(M_n = |X_{n,j,r}'| \right) = \Omega,$$

we get

$$\begin{aligned}
 L_n''(\varepsilon) &\leq \frac{1}{\tau_n'^2} \sum_{j=1}^{m(n)} \sum_{r=1}^{\ell(n)} \int_{A'_{n,j} \cap (M_n = |X'_{n,j,r}|)} |T_{j,n}|^2 d\mathbb{P} \\
 &\leq \frac{1}{\tau_n'^2} \sum_{j=1}^{m(n)} \sum_{r=1}^{\ell(n)} \int_{A'_{n,j} \cap (M_n = |X'_{n,j,r}|)} (\ell(n) |X'_{n,j,r}|^2)^2 d\mathbb{P} \\
 &\leq \frac{\ell(n)^2}{\tau_n'^2} \sum_{j=1}^{m(n)} \sum_{r=1}^{\ell(n)} \int_{(|X'_{n,j,r}| > \varepsilon \tau_n' / \ell(n))} X_{n,j,r}'^2 d\mathbb{P} \\
 &\leq \frac{\ell(n)^2}{\tau_n'^2} \sum_{j=p(n)+1}^{p(n)+n} \int_{(|X_j| > \varepsilon \tau_n' / \ell(n))} X_j^2 d\mathbb{P}
 \end{aligned}$$

We are going to complete our comparison by using a moving Lynderberg condition on the data X_j 's:

$$L'_n(\varepsilon) = \frac{\ell(n)^2}{s_n'^2} \sum_{p(n)+1 \leq j \leq p(n)+n} \int_{(|X_j| > \varepsilon s_n')} X_j^2 d\mathbb{P}. \quad (26)$$

But, by Hypothesis (Ha) and (Hab), we have $(\tau_n')^2/s_n'^2 \rightarrow 1$, as $n \rightarrow +\infty$. So, for n large enough, $(|X_j| > \varepsilon \tau_n')$ implies $(|X_j| > (\varepsilon/2)s_n')$. So, we have for n large enough, for any $\varepsilon > 0$,

$$L_n''(\varepsilon) \leq \left(\frac{s_n'}{\tau_n'}\right)^2 L'_n(\varepsilon/(2\ell(n))). \quad (27)$$

Now we may justify Theorem 6 using conditions of the real data X_j , $j \geq 1$, as announced, by summarizing the discussion above.

(B) Weak limits of Moving Partial sums using no-regrouped data.

Let us suppose that hypotheses (L), (H0), (Ha), (Hab) and (Hb) hold. The BVH hypothesis for the regrouped data $T'_{j,n}$ s is satisfied. The UAN for regrouped data is ensured by the uniform negligibility of the variances (See (19)) which itself is forced by a condition on the uniform negligibility of the variances for non-regrouped data (See (20)).

So hypotheses (L), (H0), (Ha), (Hab) and (Hb) hold and if $B'_n \rightarrow 0$, the UAN condition and the BVH are satisfied and hence, the FLL theorem for independent data $T_{j,n}$ applies and Lynderberg condition for regrouped data is forced by the Lynderberg-type condition for the non-regrouped data (17) (See (27) and (27)). Applying the

Feller-Lévy-Lynderberg (as Theorem 23 in Lo (2018), Chapter 7, section 4) to get Part (2) of Theorem 6

Also for data having $2 + \delta$ moments, for $\delta > 0$, the Lyapounov condition for regrouped data is forced by the same condition for non-regrouped data 16 (See (23) and (24)). Hence the Lyapounov theorem for the $T_{j,n}$'s applies under the Lyapounov-type condition ($A'_n \rightarrow 0$). We apply the Lyapounov theorem (as in Lo (2018)) to get Part (1) of Theorem 22, Chapter 7, Section 4). ■

5. Conclusion

The paper will be closed by Section 6 where the full computations of weak limits to the Gaussian law for moving partial sums of independent data are given. The paper, after offering a general handling moving partial sums convergence for independent data opens the rich field of such kinds of asymptotic weak laws for dependent data and their application to invariance principles and beyond to modeling in Applied statistics. The next step is to remain of these two kinds of dependence but to characterize asymptotic of random sums with not necessarily the Poissonian hypothesis on the counting process $N(\circ)$ in classical hypotheses.

6. Appendix

Proof of Theorem 1. In the partial sums beginning by $p(n) = 0$, by using Lemma 8 in Lo (2018), Theorem 1 holds for $\delta > 1$ if it does for $\delta = 1$. So it is enough to prove the theorem for $0 < \delta \leq 1$. By Lemma 9 in Lo (2018), Assumption (4) implies $s_n l' \rightarrow +\infty$ and

$$\begin{aligned} \max_{p(n)+1 \leq k \leq n+p(n)} \left(\frac{\sigma_k}{s_n l'} \right)^{2+\delta} &\leq \max_{p(n)+1 \leq k \leq n+p(n)} \frac{\mathbb{E} |X_k|^{2+\delta}}{s_n'^{2+\delta}} \\ &\leq \frac{1}{s_n'^{2+\delta}} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} |X_k|^{2+\delta} = A'_n(\delta) \rightarrow 0. \end{aligned}$$

Let us use the expansion of the characteristic functions

$$f_k(u) = \int e^{iux} dF_k(x)$$

at the order two to get for each k , $p(n) + 1 \leq k \leq n + p(n)$, as given in Lemma 6 in Lo (2018),

$$f_k(u/s_n l') = 1 - \frac{u^2}{2} \frac{\sigma_k^2}{s_n'^2} + \theta \frac{|u|^{2+\delta} \mathbb{E} |X_k|^{2+\delta}}{s_n'^{2+\delta}}, \quad |\theta| < 1. \quad (28)$$

Now the characteristic function of $S_n l'/s_n l'$ is, for $u \in \mathbb{R}$,

$$f_{S_n l'/s_n l'}(u) = \prod_{k=p(n)+1}^{n+p(n)} f_k(u/s_n l'),$$

that is

$$\log f_{S_n l'/s_n l'}(u) = \sum_{k=p(n)+1}^{n+p(n)} \log f_k(u/s_n l').$$

Now use the uniform expansion of $\log(1+u)$ at the neighborhood at 0, that is

$$\sup_{|u| \leq z} \left| \frac{\log(1+u)}{u} - 1 \right| = \varepsilon(z) \rightarrow 0. \quad (29)$$

For each k in (28), we have

$$f_k(u/s_n l') = 1 - u_{kn} \quad (30)$$

with the uniform bound

$$\begin{aligned} |u_{kn}| &\leq \frac{|u|^2}{2} \max_{p(n)+1 \leq k \leq n+p(n)} \left(\frac{\sigma_k^2}{s_n'^2} \right) + \frac{|u|^{2+\delta} \mathbb{E} |X_k|^{2+\delta}}{s_n'^{2+\delta}} \\ &= \frac{|u|^2}{2} \max_{p(n)+1 \leq k \leq n+p(n)} \left(\frac{\sigma_k^2}{s_n'^2} \right) + \frac{|u|^{2+\delta} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} |X_k|^{2+\delta}}{s_n'^{2+\delta}} =: u_n \rightarrow 0. \end{aligned}$$

Apply (29) to (30) to have

$$\log f_k(u/s_n) = -u_{kn} + \theta_n u_{kn} \varepsilon(u_n), \quad |\theta_n| < 1$$

and next

$$\begin{aligned} \log f_{S_{n'}/s_{n'}}(u) &= \sum_{k=p(n)+1}^{n+p(n)} \log f_k(u/s_n) \\ &= -\frac{u^2}{2} + |u|^{2+\delta} \theta A'_n(\delta) + \left(\frac{u^2}{2} - |u|^{2+\delta} \theta A'_n(\delta)\right) \varepsilon(u_n) \theta_n \\ &\rightarrow -u^2/2. \end{aligned}$$

We get for u fixed,

$$f_{S_{n'}/s_{n'}}(u) \rightarrow \exp(-u^2/2).$$

This completes the proof.

Proof of Theorem 2.

The proof follows the lines of that of Loève (1997). But they are extended by more details and adapted and changed in some parts. Much details were omitted. We get them back for making the proof understandable for students who just finished the measure and probability course.

Before we begin, let us establish an important property when (6) holds. Suppose that this latter holds. We want to show that there exists a sequence $\varepsilon_n \rightarrow 0$ such that $\varepsilon_n^{-2} g_n(\varepsilon_n) \rightarrow 0$ (this implying also that $\varepsilon_n^{-1} g_n(\varepsilon_n) = o(\varepsilon_n) \rightarrow 0$ and that $g_n(\varepsilon_n) = o(\varepsilon_n^2) \rightarrow 0$). To this end, let $k \geq 1$ fixed. Since $g_n(1/k) \rightarrow 0$ as $n \rightarrow \infty$, we have $0 \leq g_n(1/k) \leq k^{-3}$ for n large enough.

We will get what we want from an induction on this property. Fix $k = 1$ and denote n_1 an integer such that $0 \leq g_n(1) \leq 1^{-3}$ for $n \geq n_1$. Now we apply the same property on the sequence $\{g_n(\circ), n_1 + 1\}$ with $k = 2$. We find a $n_2 > n_1$ such that $0 \leq g_n(1/2) \leq 2^{-3}$ for $n \geq n_2$. Next we apply the same property on the sequence $\{g_n(\circ), n_2 + 1\}$ with $k = 3$. We find a $n_3 > n_2$ such that $0 \leq g_n(1/3) \leq 3^{-3}$ for $n \geq n_3$. Finally, there exists an infinite sequence of integers $n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$ such that for each $k \geq 1$, one has $0 \leq g_n(1/k) \leq k^{-3}$ for $n \geq n_k$.

Thus for each $n \geq 1$, for each $\forall k \geq 1$, there exists $k(n)$ such that

$$n_{k(n)} \leq n < n_{k(n)+1} \text{ and } 0 \leq g_n(1/k(n)) \leq k(n)^{-3}.$$

Put

$$\varepsilon_n = 1/k(n) \text{ on } n_{k(n)} \leq n < n_{k(n)+1}$$

We surely have $\varepsilon_n \rightarrow 0$ and $\varepsilon_n^{-2} g_n(\varepsilon_n)$. This is clear from

$$\begin{cases} \varepsilon_n = 1/k(n) & \text{on } n_{k(n)} \leq n < n_{k(n)+1} \\ \varepsilon_n^{-2} g_n(\varepsilon_n) = k(n)^2 (1/k(n))^3 \leq (1/k(n)) & \text{on } n_{k(n)} \leq n < n_{k(n)+1} \end{cases}.$$

Now, we are going to use

$$\varepsilon_n \rightarrow 0 \text{ and } \varepsilon_n^{-2} g_n(\varepsilon_n) \rightarrow 0. \quad (31)$$

Proof of : (6) $\implies$ (5). Suppose (6) holds. Thus there exists a sequence $(\varepsilon_n)_{n \geq 0}$ of positive numbers such that ((31)) prevails. First, we see that, for each $p(n) + 1 \leq k \leq n + p(n)$,

$$\begin{aligned} \frac{\sigma_k^2}{s_n'^2} &= \frac{1}{s_n'^2} \int x^2 dF_k(x) = \frac{1}{s_n'^2} \left\{ \int_{(|x| \geq \varepsilon_n s_n')} x^2 dF_k(x) + \int_{(|x| < \varepsilon_n s_n')} x^2 dF_k(x) \right\} \\ &\leq \frac{1}{s_n'^2} \int_{(|x| \geq \varepsilon_n s_n')} x^2 dF_k(x) + \varepsilon_n^2 \\ &\leq \frac{1}{s_n'^2} \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon_n s_n')} x^2 dF_k(x) + \varepsilon_n^2 = g(\varepsilon_n) + \varepsilon_n^2. \end{aligned}$$

It follows that

$$\max_{p(n)+1 \leq k \leq n+p(n)} \frac{\sigma_k^2}{s_n'^2} \leq g(\varepsilon_n) + \varepsilon_n^2 \rightarrow 0.$$

It remains to prove that $S_n'/s_n' \rightsquigarrow (t, \infty)$. To this end we are going to use this array of truncated random variables $\{X_{nk}, p(n) + 1 \leq k \leq n + p(n), n \geq 1\}$ defined as follows. For each fixed $n \geq 1$, define

$$X_{nk} = \begin{cases} X_k & \text{if } |X_k| \leq \varepsilon_n s_n' \\ 0 & \text{if } |X_k| > \varepsilon_n s_n' \end{cases}, p(n) + 1 \leq k \leq n + p(n).$$

Now consider summands S'_{nn} and $s_{nn}'^2$ defined by

$$S'_{nn} = \sum_{k=p(n)+1}^{n+p(n)} X_{nk} \quad \text{and} \quad s_{nn}'^2 = \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}(X_{nk} - \mathbb{E}(X_{nk}))^2.$$

We remark that for any $\eta > 0$,

$$\mathbb{P} \left(\left| \frac{S'_{nn}}{s_n'} - \frac{S'_n}{s_n'} \right| > \eta \right) \leq \mathbb{P} \left(\frac{S'_{nn}}{s_n'} \neq \frac{S'_n}{s_n'} \right)$$

and remark that

$$\begin{aligned} \left(\frac{S'_{nn}}{s_n'} \neq \frac{S'_n}{s_n'} \right) &= (\exists(p(n) + 1 \leq k \leq n + p(n)), X_{nk} \neq X_k) \\ &= (\exists(p(n) + 1 \leq k \leq n + p(n)), |X_k| > \varepsilon_n s_n') = \bigcup_{k=p(n)+1}^{n+p(n)} (|X_k| > \varepsilon_n s_n'). \end{aligned}$$

We get

$$\begin{aligned} \mathbb{P}\left(\left|\frac{S'_{nn}}{s_n'} - \frac{S'_n}{s'_n}\right| > \eta\right) &\leq \sum_{k=p(n)+1}^{n+p(n)} \mathbb{P}(|X_k| > \varepsilon_n s_n') \\ &\leq \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon_n s'_n)} dF_k(x) = \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon_n s_n)} \left\{\frac{1}{x^2}\right\} x^2 dF_k(x) \\ &\leq \left\{\frac{1}{(\varepsilon_n s'_n)^2}\right\} \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon_n s'_n)} x^2 dF_k(x) \\ &\leq \varepsilon_n^{-2} g_n(\varepsilon_n) \rightarrow 0. \end{aligned}$$

Thus S'_{nn}/s_n' and S'_n/s'_n are equivalent in probability. This implies that they have the same limit law or do not have a limit law together. So to prove that S'_n/s'_n has a limit law, we may prove that S'_{nn}/s'_n has a limit law. Next by Slutsky lemma, it will suffice to establish the limit law of S'_{nn}/s'_{nn} whenever we show that $s'_{nn}/s'_n \rightarrow 1$. We focus on this. Begin to remark that, since $\mathbb{E}(X_k) = 0$, we have the decomposition

$$0 = \mathbb{E}(X_k) = \int x dF_k(x) = \int_{(|x| \leq \varepsilon_n s_n')} x dF_k(x) + \int_{(|x| > \varepsilon_n s_n')} x dF_k(x)$$

to get that

$$\left| \int_{(|x| \leq \varepsilon_n s'_n)} x dF_k(x) \right| = \left| \int_{(|x| > \varepsilon_n s'_n)} x dF_k(x) \right|.$$

Remarking also that

$$\begin{aligned} \mathbb{E}(X_{nk}) &= \int_{(|X_k| \leq \varepsilon_n s'_n)} X_{nk} d\mathbb{P} + \int_{(|X_k| > \varepsilon_n s'_n)} X_{nk} d\mathbb{P} \\ &= \int_{(|X_k| \leq \varepsilon_n s'_n)} X_k dF_k + \int_{(|X_k| > \varepsilon_n s'_n)} 0 dF_k. \end{aligned} \tag{32}$$

Combining all that leads to

$$\begin{aligned} |\mathbb{E}(X_{nk})| &= \left| \int_{(|X_k| \leq \varepsilon_n s'_n)} X_k d\mathbb{P} \right| \\ &= \left| \int_{(|x| > \varepsilon_n s'_n)} x dF_k(x) \right| \leq \int_{(|x| > \varepsilon_n s'_n)} |x| dF_k(x) \\ &= \int_{(|x| > \varepsilon_n s'_n)} \frac{1}{|x|} x^2 dF_k(x) \leq \frac{1}{\varepsilon_n s'_n} \int_{(|x| \geq \varepsilon_n s'_n)} x^2 dF_k(x). \end{aligned}$$

Therefore

$$\frac{1}{s'_n} \sum_{k=p(n)+1}^{n+p(n)} |\mathbb{E}(X_{nk})| \leq \varepsilon_n^{-1} g(\varepsilon_n) \rightarrow 0. \quad (33)$$

With help of this let us evaluate s'_{nn}/s'_n . Notice that for each fixed $n \geq 1$, the X_{nk} are still independent. The technic used in (32) may be summarized as follows : any measurable function $g(\circ)$ such that $g(0) = 0$,

$$\mathbb{E}g(X_{nk}) = \int_{(|X_k| \leq \varepsilon_n s'_n)} g(X_{nk}) d\mathbb{P} + \int_{(|X_k| > \varepsilon_n s'_n)} g(0) d\mathbb{P} = \int_{(|X_k| \leq \varepsilon_n s'_n)} g(X_k) d\mathbb{F}. \quad (34)$$

By putting these remarks together, we obtain

$$\begin{aligned} 1 - \frac{s'^2_{nn}}{s'^2_n} &= \frac{s'^2_n - s'^2_{nn}}{s'^2_n} \\ &= \frac{1}{s'^2_n} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}X_k^2 - \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}(X_{nk} - \mathbb{E}(X_{nk}))^2 \right\} \\ &= \frac{1}{s'^2_n} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}X_k^2 - \left(\sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}(X_{nk}^2) - \mathbb{E}(X_{nk})^2 \right) \right\} \\ &= \frac{1}{s'^2_n} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}X_k^2 - \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}X_{nk}^2 + \sum_{k=p(n)+1}^{n+p(n)} (\mathbb{E}X_{nk})^2 \right\} \\ &= \frac{1}{s'^2_n} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \int X_k^2 dF_k - \sum_{k=p(n)+1}^{n+p(n)} \int_{(|X_k| \leq \varepsilon_n s'_n)} X_k^2 dF_k + \sum_{k=p(n)+1}^{n+p(n)} (\mathbb{E}X_{nk})^2 \right\}. \end{aligned}$$

This leads to

$$\begin{aligned} \left| 1 - \frac{s'^2_{nn}}{s'^2_n} \right| &= \frac{1}{s'^2_n} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \int_{(|X_k| > \varepsilon_n s'_n)} X_k^2 dF_k + \sum_{k=p(n)+1}^{n+p(n)} (\mathbb{E}X_{nk})^2 \right\} \\ &\leq \frac{1}{s'^2_n} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \int_{(|X_k| > \varepsilon_n s'_n)} X_k^2 dF_k + \sum_{k=p(n)+1}^{n+p(n)} |\mathbb{E}X_{nk}|^2 \right\} \end{aligned}$$

Finally, use the simple inequality of real numbers $(\sum_i |a_i|)^2 = \sum_i |a_i|^2 + \sum_{i \neq j} |a_i| |a_j| \geq \sum_i |a_i|^2$ and conclude from the last inequality that

$$\begin{aligned} \left| 1 - \frac{s_{nn}'^2}{s_n'^2} \right| &\leq \frac{1}{s_n'^2} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \int_{(|X_k| > \varepsilon_n s_n')} X_k^2 dF_k + \sum_{k=p(n)+1}^{n+p(n)} |\mathbb{E}(X_{nk})|^2 \right\} \\ &\leq \frac{1}{s_n'^2} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \int_{(|X_k| \geq \varepsilon_n s_n')} X_k^2 dF_k + \left(\sum_{k=p(n)+1}^{n+p(n)} |\mathbb{E}X_{nk}| \right)^2 \right\} \\ &= g(\varepsilon_n) + \left(\frac{1}{s_n'} \sum_{k=p(n)+1}^{n+p(n)} |\mathbb{E}X_{nk}| \right)^2. \end{aligned}$$

By (33) above, we arrive at

$$\left| 1 - \frac{s_{nn}'^2}{s_n'^2} \right| \leq g(\varepsilon_n) + (\varepsilon_n^{-1} g(\varepsilon_n))^2 \rightarrow 0.$$

It comes that $s_{nn}'/s_n' \rightarrow 1$. Finally, the proof of this part will derive from the limit law of S_{nn}'/s_{nn}' . We center the X_{nk} at their expectations. To prove that the new summands T_{nn}'/s_{nn}' , where

$$T_{nn}' = \sum_{k=p(n)+1}^{n+p(n)} (X_{nk} - \mathbb{E}(X_{nk})),$$

converge to $\mathcal{N}(0, 1)$, we check the Lyapounov's condition for $\delta = 1$, that is

$$\frac{1}{s_{nn}'^3} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} |X_{nk} - \mathbb{E}X_{nk}|^3 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

But we have

$$\begin{aligned} \frac{1}{s_{nn}'^3} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} |X_{nk} - \mathbb{E}X_{nk}|^3 &= \frac{1}{s_{nn}'^3} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} \left(|X_{nk} - \mathbb{E}X_{nk}| \times |X_{nk} - \mathbb{E}X_{nk}|^2 \right) \\ &\leq \frac{\mathbb{E}|X_{nk}|}{s_{nn}'^3} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} |X_{nk} - \mathbb{E}X_{nk}|^2 \\ &\quad + \frac{2}{s_{nn}'^3} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} \left(|X_{nk}| |X_{nk} - \mathbb{E}X_{nk}|^2 \right). \end{aligned}$$

Take $g(\circ) = |\circ|$ in (34) to see again that

$$\mathbb{E} |X_{nk}| = \int_{(|X_k| \leq \varepsilon_n s'_n)} |X_k| dF_k \leq \varepsilon_n s'_n.$$

We also have

$$\begin{aligned} \mathbb{E} \left(|X_{nk}| |X_{nk} - \mathbb{E} X_{nk}|^2 \right) &= \int_{(|X_k| \leq \varepsilon_n s'_n)} |X_{nk}| |X_{nk} - \mathbb{E} X_{nk}|^2 d\mathbb{P} \\ &\leq \varepsilon_n s'_n \int_{(|X_k| \leq \varepsilon_n s'_n)} |X_{nk} - \mathbb{E} X_{nk}|^2 d\mathbb{P}. \end{aligned}$$

The last formula yield

$$\begin{aligned} \frac{1}{s_{nn}^3} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} |X_{nk} - \mathbb{E} X_{nk}|^3 &\leq \frac{2\varepsilon_n s'_n}{s_{nn}^3} \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E} |X_{nk} - \mathbb{E} X_{nk}|^2 \\ &= \frac{2\varepsilon_n s'_n s_{nn}^2}{s_{nn}^3} = \varepsilon_n \frac{s'_n}{s_{nn}} \rightarrow 0. \end{aligned}$$

By Lyapounov's theorem, we have

$$\frac{T'_{nn}}{s'_{nn}} = \frac{S'_{nn} - \sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}(X_{nk})}{s'_{nn}} \rightsquigarrow \mathcal{N}(0, 1).$$

Since $s'_{nn}/s'_n \rightarrow 1$ and by (33)

$$\left| \frac{\sum_{k=p(n)+1}^{n+p(n)} \mathbb{E}(X_{nk})}{s'_{nn}} \right| \leq \frac{s'_n}{s'_{nn}} \left\{ \frac{1}{s'_n} \sum_{k=p(n)+1}^{n+p(n)} |\mathbb{E} X_{nk}| \right\} \rightarrow 0.$$

We conclude that S'_{nn}/s'_{nn} converges to $\mathcal{N}(0, 1)$.

Proof of : (5) $\implies$ (6). The convergence to $\mathcal{N}(0, 1)$ implies that for any fixed $t \in \mathbb{R}$, we have

$$\prod_{k=p(n)+1}^{n+p(n)} f_k(u/s'_n) \rightarrow \exp(-u^2/2). \quad (35)$$

We are going to use uniform expansions of $\log(1+z)$. We have

$$\lim_{z \rightarrow 0} \left| \frac{\log(1+z) - z}{z^2} \right| = \frac{1}{2},$$

this implies

$$\sup_{z \leq u} \left| \frac{\log(1+z) - z}{z^2} \right| = \varepsilon(u) \rightarrow 1/2 \text{ as } u \rightarrow 0. \quad (36)$$

Now, we use the expansion

$$f_k(u/s'_n) = 1 + \theta_k \frac{u^2 \sigma_k^2}{2s_n'^2}, \quad |\theta_k| \leq 1$$

to get that

$$\max_{p(n)+1 \leq k \leq n+p(n)} |f_k(u/s'_n) - 1| \leq \frac{u^2}{2} \max_{p(n)+1 \leq k \leq n+p(n)} \frac{\sigma_k^2}{s_n'^2} =: u_n \rightarrow 0 \quad (37)$$

and next

$$|f_k(u/s'_n) - 1|^2 = \theta_k^2 \frac{u^2 \sigma_k^2}{2s_n'^2} \times \frac{u^2 \sigma_k^2}{2s_n'^2} \leq \left[\frac{u^4}{4} \max_{p(n)+1 \leq k \leq n+p(n)} \frac{\sigma_k^2}{s_n'^2} \right] \times \frac{\sigma_k^2}{s_n'^2}.$$

This latter implies

$$\sum_{k=p(n)+1}^{n+p(n)} |f_k(u/s'_n) - 1|^2 \leq \left[\frac{u^4}{4} \max_{p(n)+1 \leq k \leq n+p(n)} \frac{\sigma_k^2}{s_n'^2} \right] =: B_n(u) \rightarrow 0.$$

By (37), we see that $\log f_k(u/s'_n)$ uniformly defined in k such that $p(n) + 1 \leq k \leq n + p(n)$ for n large enough and (35) becomes

$$\sum_{k=p(n)+1}^{n+p(n)} \log f_k(u/s'_n) \rightarrow -u^2/2$$

that is

$$\frac{u^2}{2} + \sum_{k=p(n)+1}^{n+p(n)} \log f_k(u/s'_n) \rightarrow 0.$$

Now using the uniform bound of $|f_k(u/s'_n) - 1|$ by u_n to get

$$\log f_k(u/s'_n) = f_k(u/s'_n) - 1 + \theta_{kn}(f_k(u/s'_n) - 1)^2 \varepsilon(u_n), \quad |\theta_{kn}| \leq 1$$

and then

$$\begin{aligned} \frac{u^2}{2} + \sum_{k=p(n)+1}^{n+p(n)} \log f_k(u/s'_n) &= \frac{u^2}{2} + \sum_{k=p(n)+1}^{n+p(n)} f_k(u/s'_n) - 1 + \theta_{kn}(f_k(u/s'_n) - 1)^2 \varepsilon(u_n) \\ &= \left\{ \frac{u^2}{2} - \sum_{k=p(n)+1}^{n+p(n)} (1 - f_k(u/s'_n)) \right\} \\ &\quad + \left\{ \sum_{k=p(n)+1}^{n+p(n)} \theta_{kn}(f_k(u/s'_n) - 1)^2 \right\} \varepsilon(u_n), \end{aligned}$$

with

$$\left| \left\{ \sum_{k=p(n)+1}^{n+p(n)} \theta_{kn}(f_k(u/s'_n) - 1)^2 \right\} \varepsilon(u_n) \right| \leq B_n(u) |\varepsilon(u_n)| = o(1).$$

We arrive at

$$\frac{u^2}{2} = \sum_{k=p(n)+1}^{n+p(n)} (1 - f_k(u/s'_n)) + o(1).$$

If we take the real parts, we have for any fixed $\varepsilon > 0$,

$$\begin{aligned} \frac{u^2}{2} &= \sum_{k=p(n)+1}^{n+p(n)} \int \left(1 - \cos \frac{ux}{s'_n} \right) dF_k(x) + o(1) \\ &= \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| < \varepsilon s_n^{prime})} \left(1 - \cos \frac{ux}{s'_n} \right) dF_k(x) \\ &\quad + \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon s'_n)} \left(1 - \cos \frac{ux}{s'_n} \right) dF_k(x) + o(1), \end{aligned}$$

that is

$$\frac{u^2}{2} - \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| < \varepsilon s'_n)} \left(1 - \cos \frac{ux}{s'_n} \right) dF_k(x) = \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon s'_n)} \left(1 - \cos \frac{ux}{s'_n} \right) dF_k(x) + o(1).$$

We are going to use the following fact: for all $\delta, 0 < \delta \leq 1$,

$$\sqrt{2(1 - \cos a)} \leq 2 |a/2|^\delta.$$

Apply this for $\delta = 1$ to have

$$\begin{aligned} & \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| < \varepsilon s'_n)} \left(1 - \cos \frac{ux}{s'_n}\right) dF_k(x) \\ & \leq \frac{u^2}{2s_n'^2} \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| < \varepsilon s'_n)} x^2 dF_k(x) \\ & = \frac{u^2}{2s_n'^2} \left\{ \sum_{k=p(n)+1}^{n+p(n)} \int x^2 dF_k(x) - \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon s'_n)} x^2 dF_k(x) \right\} \\ & = \frac{u^2}{2s_n'^2} \left\{ s_n'^2 - \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon s'_n)} x^2 dF_k(x) \right\} = \frac{u^2}{2} (1 - g_n(\varepsilon)). \end{aligned}$$

On the other hand

$$\begin{aligned} \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon s'_n)} \left(1 - \cos \frac{ux}{s'_n}\right) dF_k(x) & \leq 2 \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon s'_n)} dF_k(x) \\ & = 2 \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon s'_n)} \left\{ \frac{1}{x^2} \right\} x^2 dF_k(x) \\ & \leq \frac{2}{\varepsilon^2 s_n'^2} \sum_{k=p(n)+1}^{n+p(n)} \int_{(|x| \geq \varepsilon s'_n)} x^2 dF_k(x) \leq \frac{2}{\varepsilon^2}. \end{aligned}$$

By putting all this together, we have

$$\frac{u^2}{2} \leq \frac{u^2}{2} (1 - g_n(\varepsilon)) + \frac{2}{\varepsilon^2} + o(1)$$

which leads

$$\frac{u^2}{2} g_n(\varepsilon) \leq \frac{2}{\varepsilon^2} + o(1)$$

which in turns implies

$$g_n(\varepsilon) \leq \frac{2}{u^2} \left(\frac{2}{\varepsilon^2} + o(1) \right) \quad \text{for all } u \in \mathbb{R}^* \text{ and all } n \geq 1.$$

So

$$\limsup_{n \rightarrow +\infty} g_n(\varepsilon) \leq \frac{2}{u^2} \left(\frac{2}{\varepsilon^2} + o(1) \right) \quad \text{for all } u \in \mathbb{R}^*.$$

By letting $u \rightarrow +\infty$, we get

$$g_n(\varepsilon) \rightarrow 0.$$

This concludes the proof. ■

Acknowledgements. The authors would like to thank the referees and the editor for their substantial comments and suggestions, which have significantly improved the paper.

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