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Small double limit comparison with Reflecting Wentzel boundary condition

Jean Tosse Diatta [1,*], **Ibrahima Sané** [1], **Maoudo Faramba Baldé** [2], **Clément Manga** [1] **and Mamadou Saliou Bah** [2]

[1] Assane Seck University , Department of Mathematics, B.P. 523, Ziguinchor, Senegal
[2] Gamal Abdel Nasser University of Conakry, Department of Mathematics, B.P. 1147, Conakry, Guinea

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Abstract. We study a diffusion problem for Markov processes evolving in a domain with Wentzell-type boundary conditions at the borb of the domain. Our analysis focuses on the double boundary behaviour of stochastic differential equations perturbed by ε (parameter of the large deviations) and δ (a parameter of the homogeneity) which both tend to zéro. By assuming that the parameter of large deviations converges faster than the parameter of homogeneity, we establish some large-deviations-type estimates.

Key words: local time; homogenization; large deviations; stochastic differential equation; Wentzel-boundary condition.
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* Corresponding author: Maoudo Faramba Baldé (faramba88@gmail.com)
Jean Tosse Diatta: jt.d@zig.univ.sn
Ibrahima Sané: i.sane2318@zig.univ.sn
Clément Manga: cmanga@univ-zig.sn
Mamadou Saliou Bah: salioubilia@gmail.com

Résumé (Abstract in French). On étudie un probléme de diffusion pour des processus de Markov évoluant dans un domaine avec des conditions de limites de type Wentzell au bord du domaine. Notre analyse se focalise sur le comportement à double limite des équations différentielles stochastiques réfléchies perturbées par ε (paramétre des grandes déviations) et δ (paramétre de l'homogénéisation) qui tendent tous deux vers zéro. En supposons que le paramétre des grandes déviations converge plus vite que le paramétre de l'homogénéisation, nous établissons des résultats de grand déviation.

Presentations of authors.

Jean Tosse Diatta, M.Sc., is preparing a Ph.D. degree at Department of Mathematics under the supervision of Dr. Ibrahima Sané and Pr. Clémént Manga, Assane Seck University of Ziguinchor, SENEGAL.

Ibrahima Sané, PhD., is assistant Professor of Probabilities and Statistics at Department of Mathematics, Assane Seck University of Ziguinchor, SENEGAL.

Maoudo Faramba Baldé, PhD., is assistant Professor of Probabilities and Statistics at Department of Mathematics, Gamal Abdel Nasser University of Conakry, GUINEA.

Clément Manga, PhD., is associated Professor of Probabilities and Statistics at Department of Mathematics , Assane Seck University of Ziguinchor, SENEGAL

Mamadou Saliou Bah, M.Sc., is preparing a Ph.D. degree at Department of Mathematics under the supervision of Pr. Mamadou Saliou Diallo, Gamal Abdel Nasser University of Conakry, GUINEA.

1. Introduction

We consider a semi-linear partial differential equations (PDEs in the short) in the half-space $\boldsymbol{D} = \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \ : \ x_1 > 0\right\}$ with additional boundary condition on $\partial \boldsymbol{D} = \left\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \ : \ x_1 = 0\right\}$, that is, for each $\varepsilon > 0$ and $\delta_\varepsilon > 0$,

$$\begin{cases} \dfrac{\partial u^{\varepsilon,\delta_\varepsilon}}{\partial t}(t,x) = L_{\varepsilon,\delta_\varepsilon} u^{\varepsilon,\delta_\varepsilon}(t,x) + f\left(\frac{x}{\delta_\varepsilon}, u^{\varepsilon,\delta_\varepsilon}(t,x)\right) + \frac{1}{\delta_\varepsilon} e\left(\frac{x}{\delta_\varepsilon}, u^{\varepsilon,\delta_\varepsilon}(t,x)\right), \quad x \in \boldsymbol{D},\ 0 < t \\ \Gamma_{\varepsilon,\delta_\varepsilon} u^{\varepsilon,\delta_\varepsilon}(t,x) + h\left(\frac{x}{\delta_\varepsilon}, u^{\varepsilon,\delta_\varepsilon}(t,x)\right) + \dfrac{1}{\delta_\varepsilon} l\left(\frac{x}{\delta_\varepsilon}, u^{\varepsilon,\delta_\varepsilon}(t,x)\right) = 0, \quad x \in \boldsymbol{\partial D},\ 0 \leq t \\ u^{\varepsilon,\delta_\varepsilon}(0,x) = g(x) \quad x \in \partial \boldsymbol{D}, \end{cases} \tag{1}$$

where

$$L_{\varepsilon,\delta_\varepsilon} = \frac{\varepsilon}{2} \sum_{i,j=1}^{d} a_{ij}\left(\frac{x}{\delta_\varepsilon}\right) \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\varepsilon}{\delta_\varepsilon} \sum_{i=1}^{d} b_i\left(\frac{x}{\delta_\varepsilon}\right) \frac{\partial}{\partial x_i} + \sum_{i=1}^{d} c_i\left(\frac{x}{\delta_\varepsilon}\right) \frac{\partial}{\partial x_i}, \ x \in \boldsymbol{D} \tag{2}$$

and

$$\Gamma_{\varepsilon,\delta_\varepsilon} = \frac{\varepsilon}{2} \sum_{i=1}^{d} \alpha_{ij}\left(\frac{x}{\delta_\varepsilon}\right) \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\varepsilon}{\delta_\varepsilon} \sum_{i=1}^{d} \beta_i\left(\frac{x}{\delta_\varepsilon}\right) \frac{\partial}{\partial x_i} + \gamma_i\left(\frac{x}{\delta_\varepsilon}\right) \frac{\partial}{\partial x_i}, \ x \in \boldsymbol{\partial D} \tag{3}$$

Let $(L_{\varepsilon,\delta_\varepsilon}, \Gamma_{\varepsilon,\delta_\varepsilon}) - X^{x,\varepsilon,\delta_\varepsilon}$ be a diffusion process which started at x and moves the domain $\boldsymbol{D} \subseteq \mathbb{R}^d$ according to the differential generator below indexed by two parameters $\varepsilon > 0$ and $\delta_\varepsilon > 0$, and on reaching the smooth boundary $\boldsymbol{\partial D}$ of $\boldsymbol{D}$.

The large deviation principle (LDP, in short) in probability theory concerns the asymptotic behavior of tails following the law of probabilities. It formalizes the heuristic ideas of the concentration of measures and generalizes the notion of convergence in law. The pioneering works on this topic were developed by Donsker and Varadhan in Donsker *et al.*(1975). The theory of homogenization tries to understand what equations should be used at a macroscopic level, in order to approximate the behavior of physical phenomena described at a microscopic level by equations with highly oscillatory coefficients. This theory has motivated the development of various notions of weak convergence in analysis, see in particular Tartar (1990).

The combined effects of homogenization and the so-called LDP in a behaviour solution of Stochastic Differential Equation (SDE in the short) with reflecting Wentzell-boundary condition consists in computing the limit as $\varepsilon \to 0$ of $\varepsilon \log \mathbb{P}\left\{X^{x,\varepsilon,\delta_\varepsilon} \in B\right\}$, where B is a Borel subset of $C_x\left([0,T], \overline{\boldsymbol{D}}\right)$ (i.e., the set of continuous functions on $[0,T]$ with $\overline{\boldsymbol{D}}$-values which take x at zero).

This problem has been studied for diffusion processes in the case of $\boldsymbol{D} = \mathbb{R}^d$ et $\boldsymbol{\partial D} = \emptyset$ by Baldi (1991). Further, Freidlin *et al.*(1999) have generalized the work of Baldi (1991) to the case when the SDE depend on two parameters in order to explicit the rate function, and as application to wave-front propagation by comparing the relative rate between the parameters (three regimes). Later Coulibaly *et al.*(2019) have extended the work of Freidlin *et al.*(1999) with an approximation of order one at the edge of the domain when $\tau = \beta = 0$.
Inspired by above works, we propose to pursuit the work of Freidlin *et al.*(1999) to the three regime and when we have an approximation of wentzell type at the edge of the domain. More precisely, we consider only the three regime when the parameter of large deviations ε converges faster than the parameter of homogeneity δ and we prove that the family of $(L_{\varepsilon,\delta_\varepsilon}, \Gamma_{\varepsilon,\delta_\varepsilon})$-diffusion $X^{x,\varepsilon,\delta_\varepsilon}$ has a large deviations principle.

The paper is organized as follows. In section 2, we recall some adaptations by coupling homogenization and LDP for a reflected diffusion process with periodic coefficients. In section 3 we give the main result with LDP.

2. Assumptions and definitions

Now we begin with introducing some necessary notations and concepts.
Let us consider an non-empty and complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$ and by ∇ the gradient operator. We have already defined $\langle ., . \rangle$ as the standard Euclidean inner product on $\mathbb{R}^d$, and $\|.\|$ as the associated norm. Let $C_p\left(\mathbb{R}^d, \mathbb{R}^d\right)$ be the collection of continuous mapping from $\mathbb{R}^d$ into $\mathbb{R}^d$ which are periodic of period 1 in each coordinate of the argument and let $\|.\|_{C_p(\mathbb{R}^d,\mathbb{R}^d)}$ be the standard superior norm on $C\left(\mathbb{T}^d, \mathbb{R}^d\right)$ (the space of continuous mapping from $\mathbb{T}^d$ into $\mathbb{R}^d$). We consider a new defined parameter $\delta = \delta_\varepsilon$. In what follow, we state some analogue adaptations from Sane *et al.*(2021) which we will use in the rest of the paper.
We require the following assumptions:

$$\textbf{(H.1)} \quad \text{main hypothesis} \quad \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} = +\infty.$$

(H.2) L is uniformly elliptic and the matrix $a(x) = (a_{ij}(x))$ can be factored as $\sigma^{\mathrm{t}}(x)\sigma(x)$.
(H.3) The functions $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $c : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are smooth and periodic of period 1 in each variable.
(H.4) Γ is uniformly elliptic and the matrix $\alpha(x) = (\alpha_{ij}(x))$ can be factored as $\tau^{\mathrm{t}}(x)\tau(x)$.
(H.5) The functions $\tau :\ \boldsymbol{\partial D}\left(\cong \mathbb{R}^{d-1}\right) \longrightarrow \mathbb{R}^{(d-1)\times(d-1)}$, $\beta :\ \boldsymbol{\partial D}\left(\cong \mathbb{R}^{d-1}\right) \longrightarrow \mathbb{R}^{d-1}$ and $\gamma :\ \boldsymbol{\partial D}\left(\cong \mathbb{R}^{d-1}\right) \longrightarrow \mathbb{R}^d$ with $\gamma^1(x) = 1$, are smooth and periodic of period one in each direction.

Remark 1. For $x \in \boldsymbol{\partial D}$, we note $x = (0, x_2, \cdots, x_d)$, and we still set some times that $x = (x_2, \cdots, x_d)$ such that $x \in \mathbb{R}^{d-1}$. In most of the sequel, the functions τ and β are regarded as $d \times d$-matrix and d-vector respectively with the convention that $\tau^{1i} = \tau^{i1} = \beta^1 = 0$ for $1 \le i \le d$.

(Correction here)

Since $L_{\varepsilon,\delta_\varepsilon}u = 0$ inside $\boldsymbol{D}$ together with the boundary condition $\Gamma_{\varepsilon,\delta_\varepsilon}u = 0$ on $\boldsymbol{\partial D}$ determines a unique diffusion process $X^{\varepsilon,\delta_\varepsilon}$ in $\overline{\boldsymbol{D}}$ which we call the $(L_{\varepsilon,\delta_\varepsilon},\Gamma_{\varepsilon,\delta_\varepsilon})$-diffusion. We can now write the SDE satisfying by $X^{\varepsilon,\delta_\varepsilon}$. Given a quadruple of processes $\left(B^{\varepsilon,\delta_\varepsilon}, N^{\varepsilon,\delta_\varepsilon}, X^{\varepsilon,\delta_\varepsilon}, \varphi^{\varepsilon,\delta_\varepsilon}\right)$ defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ such that

1. $B^{\varepsilon,\delta_\varepsilon}$ and $N^{\varepsilon,\delta_\varepsilon}$ are d-dimensional processes such that their component processes $B_t^{\varepsilon,\delta_\varepsilon,i}$ and $N_t^{\varepsilon,\delta_\varepsilon,i}$ are square integrable continuous $\mathcal{F}_t$-martingale satisfying
   - $B_0^{\varepsilon,\delta_\varepsilon,i} = N_0^{\varepsilon,\delta_\varepsilon,i} = 0 \ \ (1 \le i \le d)$ and $N_t^{\varepsilon,\delta_\varepsilon,1} = 0$;
   - $\left\langle B^{\varepsilon,\delta_\varepsilon,i}, B^{\varepsilon,\delta_\varepsilon,j}\right\rangle_t = \delta^{ij}t \ (\ 1 \le i,j \le d)$;
   - $\left\langle B^{\varepsilon,\delta_\varepsilon,i}, N^{\varepsilon,\delta_\varepsilon,j}\right\rangle_t = 0 \ (\ 2 \le i,j \le d \ )$;
   - $\left\langle N^{\varepsilon,\delta_\varepsilon,i}, N^{\varepsilon,\delta_\varepsilon,j}\right\rangle_t = \delta^{ij}t \ (2 \le i,j \le d)$.
2. $\varphi_t^{\varepsilon,\delta_\varepsilon}$ is an $\mathcal{F}_t$-adapted and continuous increasing process such that
$$\left\{\text{point of increase of } \varphi^{\varepsilon,\delta_\varepsilon}\right\} \subset \left\{t \ge 0 : X_t^{1,\varepsilon,\delta_\varepsilon} = 0.\right\}$$
3. $X^{\varepsilon,\delta_\varepsilon}$ is an $\mathcal{F}_t$-adapted $\overline{\boldsymbol{D}}$-valued and continuous process satisfying the SDE:

$$\begin{cases} dX_t^{\varepsilon,\delta_\varepsilon} = \sqrt{\varepsilon}\sigma\left(\dfrac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dB_t^{\varepsilon,\delta_\varepsilon} + \dfrac{\varepsilon}{\delta_\varepsilon}b\left(\dfrac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dt + c\left(\dfrac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dt \\ \quad + \sqrt{\varepsilon}\tau\left(\dfrac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)dN_t^{\varepsilon,\delta_\varepsilon} + \dfrac{\varepsilon}{\delta_\varepsilon}\beta\left(\dfrac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)d\varphi_t^{\varepsilon,\delta_\varepsilon} + \gamma\left(\dfrac{X_t^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon}\right)d\varphi_t^{\varepsilon,\delta_\varepsilon}, \quad 0 < t, \\ X_t^{1,\varepsilon,\delta_\varepsilon} \ge 0, \ \varphi^{\varepsilon,\delta_\varepsilon} \text{ is continuous and increasing, } \displaystyle\int_0^t X_s^{1,\varepsilon,\delta_\varepsilon}d\varphi_s^{\varepsilon,\delta_\varepsilon} = 0, \quad 0 < t, \\ X_0^{\varepsilon,\delta_\varepsilon} = x, \end{cases} \tag{4}$$

where $X^{1,x,\varepsilon,\delta_\varepsilon}$ denotes the first component of $X^{x,\varepsilon,\delta_\varepsilon}$. We recall that $\boldsymbol{D} = \mathbb{R}_+^* \times \mathbb{R}^{d-1}$, so that $X^{x,\varepsilon,\delta}$ lives is $\overline{\boldsymbol{D}}$, that is, $X^{1,x,\varepsilon,\delta_\varepsilon}$ remains non-negative and $\phi^{\varepsilon,\delta_\varepsilon}$ increases when and only when $X^{1,x,\varepsilon,\delta_\varepsilon}$ equal to zero.
In order to simplify further the exposition, we let $\mathcal{C}\left([0;\infty[,\mathbb{R}^d\right)$ be the collection of functions that are absolutely continuous as mappings from $[0;\infty[$ into $\mathbb{R}^d$ endowed with the natural sup-norm
$$\|f\|_{[r,s]} := \sup_{r \le t \le s} \|f_t\|.$$
In particular , we set $\|f\|_{[0,T]} := \|f\|_T$ for $T > 0$.
Let $f, g \in \mathcal{C}\left([0;\infty[,\mathbb{R}^d\right)$ be such that
$$\lim_{L\to+\infty}\frac{1}{L}\int_0^L \|\dot{f}_t\|^2 dt < +\infty \quad \text{and} \quad \lim_{L\to+\infty}\frac{1}{L}\int_0^L \|\dot{g}_t\|^2 dt < +\infty.$$
Let ψ be in $\mathcal{C}\left([0;T],\overline{\boldsymbol{D}}\right)$ and let ϕ be a continuous process such that both are the unique solution of the variational representation limit problem satisfying
$$\psi_t = x + \int_0^t \mathbf{1}_{\{\psi\in\boldsymbol{D}\}}\left\{c(\psi_s) + \sigma(\psi_s)\dot{f}_s\right\}ds + \int_0^t \mathbf{1}_{\{\psi\in\boldsymbol{\partial D}\}}\left\{\gamma(\psi_s) + \tau(\psi_s)\dot{g}_s\right\}d\phi_s.$$

3. Large deviation principle (LDP)

The aim of this section is to prove that the family of $(L_{\varepsilon,\delta_\varepsilon}, \Gamma_{\varepsilon,\delta_\varepsilon})$-diffusion $X^{x,\varepsilon,\delta_\varepsilon}$ has a large deviations principle. First let us give the

Definition 1. Let $X^{x,\varepsilon,\delta_\varepsilon}$ be a $\overline{\boldsymbol{D}}$-valued random variable and let $\mathbb{P}_{\varepsilon,\delta_\varepsilon}$ denote its distribution on the Borel subsets of $\overline{\boldsymbol{D}}$, that is, $\mathbb{P}_{\varepsilon,\delta_\varepsilon}(A) = \mathbb{P}\left(X^{x,\varepsilon,\delta_\varepsilon} \in A\right)$. The family $\left\{X^{x,\varepsilon,\delta_\varepsilon}; \varepsilon \geqslant 0\right\}$ satisfies a *large deviation principle* (LDP) if there exists a lower semicontinuous function $I : \overline{\boldsymbol{D}} \longrightarrow [0, +\infty]$ such that

- for each open set $A \subseteq \overline{\boldsymbol{D}}$

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left(X^{x,\varepsilon,\delta_\varepsilon} \in A\right) \geqslant - \inf_{x \in A} I(x),$$

- for each closed set $B \subseteq \overline{\boldsymbol{D}}$

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left(X^{x,\varepsilon,\delta_\varepsilon} \in B\right) \leqslant - \inf_{x \in B} I(x).$$

Here, I is called the rate function for the LDP. A rate function is good if for each $a \in \mathbb{R}_+,\ \{x : I(x) \leqslant a\}$ is compact.

Note that by assumption on a and α, these matrix $\overline{a}$ and $\overline{\alpha}$ are strictly positive-definite. Then, letting a^{-1} and α^{-1} be its inverse matrix.
For $\kappa \in \{a, \alpha\}$, we define the norm $\|\theta\|_{\kappa^{-1}} = \sqrt{\langle \theta, \kappa^{-1}\theta \rangle}$.

$V_L :\ \overline{\boldsymbol{D}} \times \overline{\boldsymbol{D}} \longrightarrow [0, +\infty[$ is the energy function defined as

$$V_L(y,z) := V_L^1(y,z) + V_L^2(y,z)$$

where

$$V_L^1(y,z) =: \inf_{\substack{\phi \in \mathcal{C}\left([0,L];\mathbb{R}^d\right), \\ \psi(0)=x, \\ \psi(L)=y}} \int_0^L \nu^1(\dot{\psi(s)}, c(\psi(s)))ds$$

$$\begin{aligned}\nu^1(\dot{\psi(s)}, c(\psi(s))) =& \frac{1}{2}\left\langle \dot{\psi(s)} - c(\psi(s)), a^{-1}(\psi)[\dot{\psi(s)} - c(\psi(s))] \right\rangle \\ =& \frac{1}{2}\|\dot{\psi(s)} - c(\psi(s))\|^2_{a^{-1}(\psi)}\end{aligned} \tag{5}$$

and

$$V_L^2(y,z) =: \inf_{\substack{\phi \in \mathcal{C}\left([0,L];\mathbb{R}^d\right), \\ \psi(0)=x, \\ \psi(L)=y}} \int_0^L \nu^2(\dot{\psi(s)}, \gamma(\psi(s)))d\phi(s)$$

$$\begin{aligned}\nu^2(\dot{\psi(s)}, \gamma(\psi(s))) =& \frac{1}{2}\left\langle \dot{\psi(s)} - \gamma(\psi(s)), \alpha^{-1}(\psi)[\dot{\psi(s)} - \gamma(\psi(s))] \right\rangle \\ =& \frac{1}{2}\|\dot{\psi(s)} - \gamma(\psi(s))\|^2_{\alpha^{-1}(\psi)}.\end{aligned} \tag{6}$$

Let $\mathcal{J}: \mathbb{R}^d \longrightarrow [0,+\infty[$, be defined by Freidlin *et al.*(1999) as:

$$\mathcal{J} = \lim_{L\to+\infty} \frac{1}{2} V_L(0, Lz), z \in \overline{\boldsymbol{D}}.$$

From Freidlin *et al.*(1999), we know that the random family $\left\{X^{x,\varepsilon,\delta_\varepsilon}; \varepsilon \geqslant 0\right\}$ satisfies a large deviations principle with rate function defined as:

$$\inf_{\substack{\psi \in \mathcal{C}\left([0,T];\mathbb{R}^d\right),\\ \psi(0)=x,\\ \psi(T)=z}} \int_0^T \mathcal{J}(\dot{\psi(s)})ds = T\mathcal{J}\left(\frac{z-x}{T}\right).$$

We start with the following lower bound in space.

Proposition 1. *Suppose the assumptions* $(\mathbf{H}.1)$ *to* $(\mathbf{H}.5)$ *hold. For each open subset* $G \subset \overline{\boldsymbol{D}}$ *we have*

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left\{X^{x,\varepsilon,\delta_\varepsilon} \in G\right\} \geq -\inf_{z\in G} \mathcal{J}(z-x) \; .$$

Proof. Let $\psi \in \mathcal{C}\left(\left[0,1\right], \overline{\boldsymbol{D}}\right)$ satisfying $\psi_0 = x$, $\psi_1 = z$. Let us set $\hat{X}_t^{\varepsilon,\delta_\varepsilon} := \frac{1}{\sqrt{\varepsilon}}\left(X_t^{\varepsilon,\delta_\varepsilon} - \psi_t\right)$.
Now fix $z \in G$, $\eta > 0$ and let $\delta_\varepsilon^{'} > 0$ be small enough so that $\left\{z \in \overline{\boldsymbol{D}} : \|z - z'\| \leq \delta_\varepsilon^{'}\eta\right\} \subset G$. Fix also $0 < \delta_\varepsilon < \delta_\varepsilon^{'}$, then

$$\begin{aligned}
\mathbb{P}\left(X_1^{\varepsilon,\delta_\varepsilon} \in G\right) &\geq \mathbb{P}\left(\|X_1^{\varepsilon,\delta_\varepsilon} - z\| \leq \eta\delta_\varepsilon\right) \\
&\geq \mathbb{P}\left(\|X_1^{\varepsilon,\delta_\varepsilon} - \psi\|_{\mathcal{C}\left(\left[0,1\right],\overline{\boldsymbol{D}}\right)} \leq \eta\delta_\varepsilon\right) \\
&\geq \underbrace{\mathbb{P}\left(\hat{X}_t^{\varepsilon,\delta_\varepsilon}\|_{\mathcal{C}\left(\left[0,1\right],\overline{\boldsymbol{D}}\right)} \leq \eta\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)}_{:=A^\eta_{\varepsilon,\delta_\varepsilon}}
\end{aligned}$$

and $\hat{M}_s^{X^{\varepsilon,\delta_\varepsilon}} =: M_s^{X^{\varepsilon,\delta_\varepsilon}} - \frac{1}{\sqrt{\varepsilon}} \int_0^1 \psi_t d\left\langle M^{X^{\varepsilon,\delta_\varepsilon}}\right\rangle_s$ and with $\left\langle M^{X^{\varepsilon,\delta_\varepsilon}}\right\rangle_t$ denotes the quadric variation of $M_t^{X^{\varepsilon,\delta_\varepsilon}}$. We introduce the measure $\hat{\mathbb{P}}$ on $\left(\Omega, \mathcal{F}\right)$ defined as:

$$\frac{\mathrm{d}\hat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} := \exp\left(\frac{1}{\sqrt{\varepsilon}}\left\langle\theta, M_t^{\frac{1}{\delta}X^{x,\varepsilon,\delta_\varepsilon}}\right\rangle - \frac{1}{2\varepsilon}\left\langle\theta, \left\langle M_t^{\frac{1}{\delta}X^{x,\varepsilon,\delta_\varepsilon}}\right\rangle_t\theta\right\rangle\right).$$

It follows that

$$\mathbb{P}\left(A^\eta_{\varepsilon,\delta_\varepsilon}\right) = \hat{\mathbb{E}}\left[\mathbf{1}_{\left\{A^\eta_{\varepsilon,\delta_\varepsilon}\right\}} e^{\frac{1}{\sqrt{\varepsilon}}\int_0^1 d\left\langle\psi, M^{X^{\varepsilon,\delta_\varepsilon}}\right\rangle_s - \frac{1}{2\varepsilon}\int_0^1\left\langle\psi,\left\langle M_t^{X^{\varepsilon,\delta_\varepsilon}}\right\rangle_t\psi\right\rangle dt}\right] \tag{7}$$

Notice that

$$A^{\eta}_{\varepsilon,\delta_\varepsilon} = \left\{ \sup_{0\leq t\leq 1} \| \frac{X^{\varepsilon,\delta_\varepsilon}}{\delta_\varepsilon} - \frac{\psi_t}{\delta_\varepsilon} \| \leq \eta \right\}$$

and on this set, we have $\frac{1}{2}\int_0^1 \left\langle \psi, \left\langle M_t^{X^{\varepsilon,\delta_\varepsilon}} \right\rangle_t \psi \right\rangle dt \leq \hat{V}\left(\psi,\varepsilon,\eta\right)$, where

$$\begin{aligned} \hat{V}\left(\psi,\varepsilon,\eta\right) :=& \sup_{\left\{ \|\rho\|_{\mathcal{C}\left(\left[0,1\right],\overline{\boldsymbol{D}}\right)} \leq \eta \right\}} \frac{1}{2}\int_0^1 \|\dot{\psi}_s - B_1^{\varepsilon,\delta_\varepsilon}\left(\frac{\psi_s}{\delta_\varepsilon} + \rho(s)\right)\|^2_{a^{-1}\left(\frac{\psi_s}{\delta_\varepsilon}+\rho(s)\right)} \\ &+ \sup_{\left\{ \|\rho\|_{\mathcal{C}\left(\left[0,1\right],\overline{\boldsymbol{D}}\right)} \leq \eta \right\}} \frac{1}{2}\int_0^1 \|\dot{\psi}_s - B_2^{\varepsilon,\delta_\varepsilon}\left(\frac{\psi_s}{\delta_\varepsilon} + \rho(s)\right)\|^2_{\alpha^{-1}\left(\frac{\psi_s}{\delta_\varepsilon}+\rho(s)\right)} \end{aligned}$$

where

$$B_1^{\varepsilon,\delta_\varepsilon}(z) := \frac{\varepsilon}{\delta_\varepsilon} b(z) + c(z) \quad , \quad B_2^{\varepsilon,\delta_\varepsilon}(z) := \frac{\varepsilon}{\delta_\varepsilon}\beta(z) + \gamma(z).$$

As in Freidlin *et al.*(1999) (Young's inequality), there exists constants m_1, m_2, k_1 and $k_2 > 0$ such that for all $\tilde{\eta} > 0$,

$$\begin{aligned} &\frac{1}{2}\|\dot{\psi}_t - B_1^{\varepsilon,\delta_\varepsilon}\left(\frac{\psi_t}{\delta_\varepsilon} + \rho(t)\right)\|^2_{a^{-1}\left(\frac{\psi_t}{\delta_\varepsilon}+\rho(t)\right)} \\ &\qquad \leq \frac{1}{2}\left(1+k_1\right)\left(1+\tilde{\eta}^{-1}\right)\|\dot{\psi}_s - c\left(\frac{\psi_t}{\delta_\varepsilon}\right)\|^2_{a^{-1}\left(\frac{\psi_t}{\delta_\varepsilon}\right)} + W^1\left(\varepsilon,\eta,\tilde{\eta}\right) \\ &\frac{1}{2}\|\dot{\psi}_t - B_2^{\varepsilon,\delta_\varepsilon}\left(\frac{\psi_t}{\delta_\varepsilon} + \rho(t)\right)\|^2_{a^{-1}\left(\frac{\psi_t}{\delta_\varepsilon}+\rho(t)\right)} \\ &\qquad \leq \frac{1}{2}\left(1+m_1\right)\left(1+\tilde{\eta}^{-1}\right)\|\dot{\psi}_s - \gamma\left(\frac{\psi_t}{\delta_\varepsilon}\right)\|^2_{\alpha^{-1}\left(\frac{\psi_t}{\delta_\varepsilon}\right)} + W^2\left(\varepsilon,\eta,\tilde{\eta}\right) \end{aligned}$$

with

$$\begin{aligned} W^1\left(\varepsilon,\eta,\tilde{\eta}\right) &:= \sup_{\left\{z,z'\in\overline{\boldsymbol{D}},\ \|z-z'\|\leq\eta\right\}} \frac{1}{2}\left(1+k_1\eta\right)\left(1+\tilde{\eta}^{-1}\right)k_2\left\{\frac{\varepsilon}{\delta_\varepsilon}\|b(z)\| + \|c(z)-c(z')\|\right\} \\ W^2\left(\varepsilon,\eta,\tilde{\eta}\right) &:= \sup_{\left\{z,z'\in\overline{\boldsymbol{D}},\ \|z-z'\|\leq\eta\right\}} \frac{1}{2}\left(1+m_1\eta\right)\left(1+\tilde{\eta}^{-1}\right)m_2\left\{\frac{\varepsilon}{\delta_\varepsilon}\|\beta(z)\| + \|\gamma(z)-\gamma(z')\|\right\}. \end{aligned}$$

Observe that $W^1\left(\varepsilon,\eta,\tilde{\eta}\right)$ and $W^1\left(\varepsilon,\eta,\tilde{\eta}\right)$ do not depend on our choice of x or $z \in \overline{\boldsymbol{D}}$ and

$$W\left(\varepsilon,\eta,\tilde{\eta}\right) := W^1\left(\varepsilon,\eta,\tilde{\eta}\right) + W^2\left(\varepsilon,\eta,\tilde{\eta}\right)$$

on our choice $\psi \in \mathcal{C}\left(\left[0,1\right],\overline{\boldsymbol{D}}\right)$ and that $\lim_{\varepsilon\to 0,\eta\to 0} W\left(\varepsilon,\eta,\tilde{\eta}\right) = 0$.

From (7) we have

$$\begin{aligned}\mathbb{P}\left(A^{\eta}_{\varepsilon,\delta_\varepsilon}\right) &\geq \exp\left(-\hat{V}\left(\psi,\varepsilon,\eta\right)\right)\times\exp\frac{\hat{\mathbb{E}}\left[\exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^1\psi(s)dM_s^{X^{\varepsilon,\delta_\varepsilon}}\right\}\mathbf{1}_{\left\{A^{\eta}_{\varepsilon,\delta_\varepsilon}\right\}}\right]\hat{\mathbb{P}}\left(A^{\eta}_{\varepsilon,\delta_\varepsilon}\right)}{\hat{\mathbb{P}}\left(A^{\eta}_{\varepsilon,\delta_\varepsilon}\right)}\\ &\geq \exp\left(-\hat{V}\left(\psi,\varepsilon,\eta\right)\right)\times\underbrace{\exp\left(-\frac{1}{\sqrt{\varepsilon}}\frac{\hat{\mathbb{E}}\left[\int_0^1\psi(s)\mathbf{1}_{\left\{A^{\eta}_{\varepsilon,\delta_\varepsilon}\right\}}dM_s^{X^{\varepsilon,\delta_\varepsilon}}\right]}{\hat{\mathbb{P}}\left(A^{\eta}_{\varepsilon,\delta_\varepsilon}\right)}\right)\times\hat{\mathbb{P}}\left(A^{\eta}_{\varepsilon,\delta_\varepsilon}\right)}_{\text{Jensen's inequality}}\end{aligned}\tag{8}$$

and

$$\mathbb{P}\left(A^{\eta}_{\varepsilon,\delta_\varepsilon}\right)\geq\exp\left(-\hat{V}\left(\psi,\varepsilon,\eta\right)\right)\times\underbrace{\exp\left[-\frac{K^{1/k_5}}{\sqrt{\varepsilon}}\frac{\sqrt{2\hat{V}\left(\psi,\varepsilon,\eta\right)}}{(\tilde{W}_{\varepsilon,\delta_\varepsilon})^{1/k_5}}\right]\times\tilde{W}_{\varepsilon,\delta_\varepsilon}}_{\text{Burkholder-Davis-Gundy inequality}}\tag{9}$$

where, similarly as in Lemma 4.4 of Freidlin *et al.*(1999).

$$\tilde{W}_{\varepsilon,\delta_\varepsilon}:=\left(\frac{2\min\left(1,\frac{\delta_\varepsilon}{\sqrt{\varepsilon}}\right)}{\sqrt{2\pi e}}\right)^{k_5}.$$

Now, we put everything together, rescale the integral on the right of (7), and $\psi\in\mathcal{C}\left(\left[0,1\right],\overline{\boldsymbol{D}}\right)$ satisfying $\psi_0=x$, $\psi_1=z$. Then we have

$$\begin{aligned}&\mathbb{P}\left(A^{\eta}_{\varepsilon,\delta_\varepsilon}\right)\\ &\geq e^{\left(\frac{1}{\varepsilon}\left[(1+k_1\eta)(1+\tilde{\eta})\delta_\varepsilon V_{\frac{1}{\delta_\varepsilon}}+W(\varepsilon,\eta,\tilde{\eta})\right]\right)}\times e^{\left\{-\frac{K^{1/k_5}}{\sqrt{\varepsilon}}\cdot\frac{\sqrt{2(1+k_1\eta)(1+\tilde{\eta})\delta_\varepsilon V_{\frac{1}{\delta_\varepsilon}}+W(\varepsilon,\eta,\tilde{\eta})}}{\left(\tilde{W}_{\varepsilon,\delta_\varepsilon}\right)^{1/k_5}}\right\}}\times\tilde{W}_{\varepsilon,\delta_\varepsilon}.\end{aligned}\tag{10}$$

Thus letting consecutively ε, δ_ε, η and then $\tilde{\eta}$ tend to zero in that order, we get

$$\liminf_{\varepsilon\to0}\log\mathbb{P}\left\{X_1^{\varepsilon,\delta_\varepsilon}\in G\right\}\geq-\inf_{z\in G}\mathcal{J}\left(z-x\right).$$

The upper bound

Next let's prove large deviations upper bound. We shall do this by appealing to the bounds of Norris *et al.*(1991). For each $\varepsilon>0$, let $L^*_{\varepsilon,\delta_\varepsilon}$ be the adjoint of $L_{\varepsilon,\delta_\varepsilon}$ and $\Gamma^*_{\varepsilon,\delta_\varepsilon}$ be the adjoint of $\Gamma_{\varepsilon,\delta_\varepsilon}$ with respect to Lebesgue measure on $\left(\overline{\boldsymbol{D}},\mathcal{B}\left(\overline{\boldsymbol{D}}\right)\right)$ and let $\{\boldsymbol{P}^\varepsilon_t:t\geq0\}$ be the semigroup on $\mathcal{B}\left(\overline{\boldsymbol{D}}\right)$ the Banach space of bounded and measurable functions defined by the adjoints or operators equipped with the Wentzhel boundary condition. Then, by virtue of the positivity and periodicity of the coefficients, for each $\varepsilon>0$ there is a

$$\left(\boldsymbol{P}^\varepsilon_t\phi\right)(z)=\int_{\overline{\boldsymbol{D}}}\boldsymbol{P}^\varepsilon(t,z,y)\phi(y)dy,t\geq0,z\in\overline{\boldsymbol{D}},\phi\in\mathcal{B}\left(\overline{\boldsymbol{D}}\right).$$

Thus for any $A \in \mathcal{B}\left(\overline{\boldsymbol{D}}\right)$ the σ-algebra of all Borel subsets of $\overline{\boldsymbol{D}}$,

$$\mathbb{P}\left(X_t^{\varepsilon,\delta_\varepsilon} \in A\right) = \int_{A/\delta_\varepsilon} \boldsymbol{P}^{\varepsilon}((\sqrt{\varepsilon}/\delta_\varepsilon)^2 t, z, \frac{x}{\delta_\varepsilon})dz, t \geq 0, A \in \mathcal{B}\left(\overline{\boldsymbol{D}}\right). \tag{11}$$

Let's begin by reconciling the notation of Norris *et al.*(1991) with ours. Fix also $\phi \in \mathcal{B}\left(\overline{\boldsymbol{D}}\right)$ and define, for each $\varepsilon > 0$, $\phi_t := \boldsymbol{P}_t^{\varepsilon}\phi$

$$\begin{aligned}
&\int_{\boldsymbol{D}} f(z)\dot{\phi}_t(z)dz \\
&= -\boldsymbol{D} f(z)\left(L^*_{\varepsilon,\delta_\varepsilon}\phi_t\right)(z)dz - \int_{\partial \boldsymbol{D}} f(z)\left(\Gamma^*_{\varepsilon,\delta_\varepsilon}\phi_t\right)(z)dz \\
&= -\int_{\boldsymbol{D}}\left(L^*_{\varepsilon,\delta_\varepsilon} f\right)(z)\phi_t dz - \int_{\partial \boldsymbol{D}}\left(\Gamma^*_{\varepsilon,\delta_\varepsilon} f\right)(z)\phi_t dz \\
&= \int_{\boldsymbol{D}}\left\{\langle \nabla f, \nabla \phi_t\rangle_{\underline{\mathfrak{a}}^{\varepsilon,\delta_\varepsilon}} - f\left\langle \underline{\mathfrak{b}}^{\varepsilon,\delta_\varepsilon}, \nabla\phi_t\right\rangle_{\underline{\mathfrak{a}}^{\varepsilon,\delta_\varepsilon}} - \phi_t\left\langle \hat{\underline{\mathfrak{b}}}^{\varepsilon,\delta_\varepsilon}, \nabla f\right\rangle_{\underline{\mathfrak{a}}^{\varepsilon,\delta_\varepsilon}} - \underline{\mathfrak{c}}^{\varepsilon,\delta_\varepsilon} f\phi_t\right\}(z)dz \\
&\quad + \int_{\partial \boldsymbol{D}}\left\{\langle \nabla f, \nabla \phi_t\rangle_{\underline{\alpha}^{\varepsilon,\delta_\varepsilon}} - f\left\langle \underline{\beta}^{\varepsilon,\delta_\varepsilon}, \nabla\phi_t\right\rangle_{\underline{\alpha}^{\varepsilon,\delta_\varepsilon}} - \phi_t\left\langle \hat{\underline{\beta}}^{\varepsilon,\delta_\varepsilon}, \nabla f\right\rangle_{\underline{\alpha}^{\varepsilon,\delta_\varepsilon}} - \underline{\gamma}^{\varepsilon,\delta_\varepsilon} f\phi_t\right\}.
\end{aligned} \tag{12}$$

The underlined coefficients are explicitly chosen taking into account (4). As in Norris *et al.*(1991) the coefficients $\underline{\mathfrak{b}}^{\varepsilon,\delta_\varepsilon}$, $\hat{\underline{\mathfrak{b}}}^{\varepsilon,\delta_\varepsilon}$, $\underline{\beta}^{\varepsilon,\delta_\varepsilon}$, $\hat{\underline{\beta}}^{\varepsilon,\delta_\varepsilon}$ grow like $\frac{\delta_\varepsilon}{\varepsilon}$ when ε, δ_ε tend to zero. We will take

$$\begin{aligned}
&\underline{\mathfrak{a}}^{\varepsilon,\delta_\varepsilon}(z) = \frac{1}{2}a(z),\ \underline{\mathfrak{c}}^{\varepsilon,\delta_\varepsilon}(z) = -\frac{1}{2}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)\operatorname{div}\left(c + \frac{\delta_\varepsilon}{\varepsilon}b\right)(z) \\
&\underline{\mathfrak{b}}^{\varepsilon,\delta_\varepsilon}(z) = a^{-1}\left(b + \frac{\delta_\varepsilon}{\varepsilon}c\right)(z),\ \hat{\underline{\mathfrak{b}}}^{\varepsilon,\delta_\varepsilon}(z) = -a^{-1}\left(b + \frac{\delta_\varepsilon}{\varepsilon}c\right)(z) + a^{-1}\left(\operatorname{div}(a)\right)^*(z)
\end{aligned} \tag{13}$$

and thus

$$\begin{aligned}
&\underline{\alpha}^{\varepsilon,\delta_\varepsilon}(z) = \frac{1}{2}\alpha(z),\ \underline{\gamma}^{\varepsilon,\delta_\varepsilon}(z) = -\frac{1}{2}\left(\frac{\delta_\varepsilon}{\varepsilon}\right)\operatorname{div}\left(\gamma + \frac{\delta_\varepsilon}{\varepsilon}\beta\right)(z) \\
&\underline{\beta}^{\varepsilon,\delta_\varepsilon}(z) = \alpha^{-1}\left(\beta + \frac{\delta_\varepsilon}{\varepsilon}\gamma\right)(z),\ \hat{\underline{\beta}}^{\varepsilon,\delta_\varepsilon}(z) = -\alpha^{-1}\left(\beta + \frac{\delta_\varepsilon}{\varepsilon}\gamma\right)(z) + \alpha^{-1}\left(\operatorname{div}(\alpha)\right)^*(z).
\end{aligned} \tag{14}$$

Next, for each ε, $\delta_\varepsilon > 0$ and $t < 0$, we define the quasi potential

$$\begin{aligned}
\mathcal{E}^{\varepsilon,\delta_\varepsilon}(t,z,y) :=& \frac{1}{4}\inf_{\substack{\psi \in \mathcal{C}\left([t,0];\overline{\boldsymbol{D}}\right),\\ \psi(t)=y,\\ \psi(0)=z}} \int_t^0 \|\dot{\psi}_t - \left(\underline{\mathfrak{a}}^{\varepsilon,\delta_\varepsilon}\left[\underline{\mathfrak{b}}^{\varepsilon,\delta_\varepsilon} - \hat{\underline{\mathfrak{b}}}^{\varepsilon,\delta_\varepsilon}\right]\right)(\psi_t)\|_{(a^{\varepsilon,\delta_\varepsilon})^{-1}(\psi_s)}ds \\
&+ \frac{1}{4}\inf_{\substack{\psi \in \mathcal{C}\left([t,0];\overline{\boldsymbol{D}}\right),\\ \psi(t)=y,\\ \psi(0)=z}} \int_t^0 \|\dot{\psi}_t - \left(\underline{\alpha}^{\varepsilon,\delta_\varepsilon}\left[\underline{\beta}^{\varepsilon,\delta_\varepsilon} - \hat{\underline{\beta}}^{\varepsilon,\delta_\varepsilon}\right]\right)(\psi_t)\|_{(\alpha^{\varepsilon,\delta_\varepsilon})^{-1}(\psi_s)}d\phi_s.
\end{aligned} \tag{15}$$

Thanks to these preliminary results, we are now in the position to investigate the upper bound in space.

Proposition 2. *Suppose the assumptions* $(\mathbf{H}.1)$ *to* $(\mathbf{H}.5)$ *holds. For each closed subset* $F \subset \overline{\boldsymbol{D}}$ *we have*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left\{X^{x,\varepsilon,\delta_\varepsilon} \in G\right\} \leq -\inf_{z\in F} \mathcal{J}\left(z-x\right).$$

Proof. First, let us observe that

$$\begin{aligned}\boldsymbol{P}^{\varepsilon,\delta_\varepsilon} \leq\ & \frac{K}{-t}\Big(1 - t\sup_{z\in\overline{\boldsymbol{D}}}\Big\{\|\underline{b}^{\varepsilon,\delta_\varepsilon}\|^2_{\underline{a}^{\varepsilon,\delta_\varepsilon}} + \|\hat{\underline{b}}^{\varepsilon,\delta_\varepsilon}\|^2_{\underline{a}^{\varepsilon,\delta_\varepsilon}} + \|\underline{c}^{\varepsilon,\delta_\varepsilon}\|^2_{\underline{a}^{\varepsilon,\delta_\varepsilon}} + \|\underline{\beta}^{\varepsilon,\delta_\varepsilon}\|^2_{\underline{\alpha}^{\varepsilon,\delta_\varepsilon}} \\ & + \|\hat{\underline{\beta}}^{\varepsilon,\delta_\varepsilon}\|^2_{\underline{\alpha}^{\varepsilon,\delta_\varepsilon}} + \|\underline{\gamma}^{\varepsilon,\delta_\varepsilon}\|^2_{\underline{\alpha}^{\varepsilon,\delta_\varepsilon}} + \mathcal{E}^{\varepsilon,\delta_\varepsilon}(t,z,y)\Big\}\Big) \\ & \times\exp\left\{\left(-t\sup_{z\in\overline{\boldsymbol{D}}}\left\{\frac{1}{4}\|\underline{b}^{\varepsilon,\delta_\varepsilon}+\hat{\underline{b}}^{\varepsilon,\delta_\varepsilon}\|^2 + \frac{1}{4}\|\underline{\beta}^{\varepsilon,\delta_\varepsilon}+\hat{\underline{\beta}}^{\varepsilon,\delta_\varepsilon}\|^2 + \underline{c}^{\varepsilon,\delta_\varepsilon} + \underline{\gamma}^{\varepsilon,\delta_\varepsilon}\right\} - \mathcal{E}^{\varepsilon,\delta_\varepsilon}(t,z,y)\right)\right\}. \end{aligned} \tag{16}$$

Taking into account (13) and (14) we have,

$$\boldsymbol{P}^{\varepsilon,\delta_\varepsilon} \leq K\left(\frac{1+\frac{\delta_\varepsilon}{\varepsilon}\left(\frac{\delta_\varepsilon}{\varepsilon}+1\right)t + \mathcal{E}^{\varepsilon,\delta_\varepsilon}(-t,z,y)}{t}\right)^{\frac{d}{2}} \times\exp\left\{K\Big(\frac{\delta_\varepsilon}{\varepsilon}+1\Big)t - \mathcal{E}^{\varepsilon,\delta_\varepsilon}(-t,z,y)\right\}. \tag{17}$$

From (11), we have

$$\begin{aligned}\mathbb{P}\left(X_t^{\varepsilon,\delta_\varepsilon}\in A\right) &= \int_{A/\delta_\varepsilon}\boldsymbol{P}^{\varepsilon}((\sqrt{\varepsilon}/\delta_\varepsilon)^2 t, z, \frac{x}{\delta_\varepsilon})dz \\ &\leq \delta_\varepsilon^{-d}\underbrace{\int_{A/\delta_\varepsilon}\boldsymbol{P}^{\varepsilon}((\sqrt{\varepsilon}/\delta_\varepsilon)^2 t, \frac{z}{\delta_\varepsilon}, \frac{x}{\delta_\varepsilon})dz}_{\text{by scaling property}}.\end{aligned} \tag{18}$$

Then we have by (17),

$$\begin{aligned}\mathbb{P}\left(X_t^{\varepsilon,\delta_\varepsilon}\in A\right) \leq K\delta_\varepsilon^{-d}\int_A &\left(\frac{1+\frac{\delta_\varepsilon}{\varepsilon}\left(\frac{\delta_\varepsilon}{\varepsilon}+1\right)(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon})^2 + \mathcal{E}^{\varepsilon,\delta_\varepsilon}(-(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon})^2,\frac{z}{\delta_\varepsilon},\frac{x}{\delta_\varepsilon})}{(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon})^2}\right)^{\frac{d}{2}} \\ &\times\exp\left\{K\Big(\frac{\delta_\varepsilon}{\varepsilon}+1\Big)(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon})^2 - \mathcal{E}^{\varepsilon,\delta_\varepsilon}(-(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon})^2,\frac{z}{\delta_\varepsilon},\frac{x}{\delta_\varepsilon})\right\}.\end{aligned} \tag{19}$$

Thus on can see that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon\mathcal{E}^{\varepsilon,\delta_\varepsilon}\left(-(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon})^2,\frac{z}{\delta_\varepsilon},\frac{x}{\delta_\varepsilon}\right) &= \inf\left\{\lim_{\varepsilon\to 0}\delta_\varepsilon V_{\frac{1}{\delta_\varepsilon}}\left(\frac{z}{\delta_\varepsilon},\frac{x}{\delta_\varepsilon},\mathsf{T}\right) : \lim_{L\to+\infty}\int_0^L \mathsf{T}^2 ds\right\} \\ &= \mathcal{J}\left(z-x\right).\end{aligned}$$

Morever, this limits is uniform for z in any compact subset in $\overline{\boldsymbol{D}}$.
Another remark is that for some positive $K_1^{'}$ and $K_2^{'}$,

$$\varepsilon\mathcal{E}^{\varepsilon,\delta_\varepsilon}\left(-(\frac{\sqrt{\varepsilon}}{\delta_\varepsilon})^2,\frac{z}{\delta_\varepsilon},\frac{x}{\delta_\varepsilon}\right) \geq K_1^{'}|z-x\|^2 - K_2^{'}.$$

Finally, by a Laplace-type argument we get,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\left\{X^{x,\varepsilon,\delta_\varepsilon}\in G\right\} &\leq -\inf_{z\in F}\mathcal{J}\left(z-x\right)+\lim_{\varepsilon\to 0}K_1^{'}\left[\frac{\varepsilon}{\delta_\varepsilon}+\left(\frac{\varepsilon}{\delta_\varepsilon}\right)^2\right] \\ &\leq -\inf_{z\in F}\mathcal{J}\left(z-x\right).\end{aligned} \tag{20}$$

4. Conclusion

We have used the general expression of a function for a family of stochastic equations with a second-order approximation at the edge of the domain to generalize the work of Freidlin *et al.*(1999) with respect to three regime. We present the main result in Section 3, that is, the large deviation principe for the family of $(L_{\varepsilon,\delta_\varepsilon},\Gamma_{\varepsilon,\delta_\varepsilon})$-diffusion $X^{x,\varepsilon,\delta_\varepsilon}$. Since we are interested in the process trajectory with probability measures, such results may have implications for both random optimization and statistics when studying trajectories for density estimates or regression problems.

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