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# Theoretical analysis and improvements in cubic transmutations of probability distributions

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**Abstract.** In statistics, processed data are becoming increasingly complex, and classical probability distributions are limited in their ability to model them. This is why, to better model data, extensive work has been conducted on extending classical probability distributions. Generally, this extension is achieved by transforming the cumulative distribution function of a baseline distribution through the addition of one or more parameters to enhance its flexibility. (See page 3868 for the full abstract).

**Key words:** Cubic transmutation; Pareto distribution; parameter estimation; maximum likelihood; numerical optimization.
**AMS 2010 Mathematics Subject Classification Objects :** 60E05; 62E15; 62F10; 62F99; 62P99.

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**Full Abstract in English**. In statistics, processed data are becoming increasingly complex, and classical probability distributions are limited in their ability to model them. This is why, to better model data, extensive work has been conducted on extending classical probability distributions. Generally, this extension is achieved by transforming the cumulative distribution function of a baseline distribution through the addition of one or more parameters to enhance its flexibility. Cubic transmutation (CT) is one of the most popular methods for such extensions. However, CT does not have a unique definition because different approaches for CT have been proposed in the literature but are yet to be compared. The main goal of this paper is to compare these different approaches from both theoretical and empirical viewpoints. We study the relationships between the different approaches and we propose modified versions based on the extension of parameter ranges. The results are illustrated using Pareto distribution as baseline distribution.

**Résumé (Abstract in French)**. En statistique, les données traitées deviennent de plus en plus complexes et les lois de probabilité classiques sont limitées dans leur capacité à les modéliser. C'est pourquoi, afin de mieux modéliser les données, des travaux approfondis ont été menés sur l'extension des lois de probabilité classiques. Généralement, cette extension est réalisée en transformant la fonction de répartition d'une distribution de base grâce à l'ajout d'un ou plusieurs paramètres pour améliorer sa flexibilité. La transmutation cubique est l'une des méthodes les plus populaires pour de telles extensions. Cependant, elle n'a pas de définition unique car différentes approches de transmutation cubique ont été proposées dans la littérature mais doivent encore être comparées. L'objectif de cet article est de comparer ces différentes approches à la fois théoriquement et empiriquement. Nous étudions les relations entre les différentes approches et proposons des versions modifiées basées sur l'extension des plages de paramètres. Les résultats sont illustrés à en utilisant la loi de Pareto comme distribution de base.

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## 1. Introduction

In statistics, the data to be processed are becoming increasingly complex, and classical probability distributions are limited in their ability to model them. This is why, in order to better model the data, considerable work has focused on extending classical probability distributions. Generally, this extension is achieved by transforming a given baseline cumulative distribution function (*cdf*) into another by adding one or more new parameters to enhance its flexibility. This is how, over the past few decades, scientific research has seen the emergence of new families of probability distributions. The reader can consult references such as Ahmad et al. (2019), Ahmad et al. (2022) and Tahir and Cordeiro (2016).

Among the techniques for extending probability distributions, transmutation is one of the most used. The first form of transmutation is the quadratic transmutation introduced by Shaw and Buckley (2009). It consists of transforming a baseline *cdf* $G(x)$ (that can depend on one or more parameters) in the form of a new *cdf* defined for all $x \in \mathbb{R}$ by

$$F_S(x) = (1 + \lambda)G(x) - \lambda G^2(x), \tag{1}$$

where $\lambda$ is an additional parameter such that $|\lambda| \leqslant 1$. It can be noticed that the value $\lambda = 0$ enables to get the baseline *cdf* $G(x)$. Several authors (see Tahir and Cordeiro (2016), Dey et al. (2021), Imliyangba et al. (2021)) have applied the formula (1) to obtain quadratic transmuted distributions and verified on real data that these latter fit the data better than the baseline distributions. Later, Granzotto et al. (2017) showed that the formula (1) represents the *cdf* of a mixture of order statistics. But this quadratic transmutation does not adapt to certain complex data (see Rahman et al. (2020a) and Saha et al. (2024)). Thus, in order to obtain more flexibility, Granzotto et al. (2017) then extended it to develop cubic transmutation (*CT*) through the new *cdf*

$$F_G(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1)G^2(x) + (1 - \lambda_2)G^3(x), \tag{2}$$

where $\lambda_1$ and $\lambda_2$ are additional parameters such that $0 \leqslant \lambda_1 \leqslant 1$ and $-1 \leqslant \lambda_2 \leqslant 1$. They applied the *CT* to Weibull and log-logistic distributions on real data and found that the cubic transmutations presented better fits to the data than the quadratic transmutation and the baseline distribution.

Following Granzotto et al. (2017), several authors have developed other cubic transmutation formulas. We can mention AL-Kadim and Mohammed (2017), Rahman et al. (2018a), Rahman et al. (2018b), Rahman et al. (2019b) and Rahman et al. (2023). For each formula, the authors verified using real data that the proposed cubic transmutation fit the data better than the quadratic transmutation. Some of the cubic transmutation formulas have been compared to each other but all these six cubic transmutation formulas have yet to be all compared with each other. Some have been compared individually to the formulas of Granzotto et al. (2017) and/or AL-Kadim and Mohammed (2017) for Weibull, Pareto, log-logistic, power function, Kumaraswamy distributions (see Aslam et al. (2020), Rahman et al. (2019a), Rahman et al. (2021), Rahman et al. (2024), Hafeez et al. (2023), Tushar et al. (2024)). Sakthivel and Vidhya (2023) applied four of these six cubic transmutation formulas on Rayleigh distribution, gave their statistical properties and compared the four different formulas for fitting three datasets. However, the four formulas were not simultaneously compared with each other on the same data set. Moreover, the theoretical aspects of the comparison were also not considered.

The objective of this paper is to provide a comparative analysis of different cubic transmutations. The paper is organized as follows. In Section 2, we review the state of the art regarding cubic transmutation formulas. Section 3 presents a critical analysis of the various formulas, focusing particularly on the parameter ranges. This analysis will lead to the proposal of modified parameter ranges that extend the obtained distributions. We will also examine the formulas to determine if some of the studied families of distributions can be obtained as subfamilies of the others. Section 4 presents an illustration of our results by comparing the performances of different cubic transmuted Pareto distributions in fitting real data in R software (R Core Team, 2024). The paper concludes with a discussion and final remarks in Section 5.

## 2. Cubic transmutation: a review of existing formulas

Let $x \in \mathbb{R}$ and $G(x)$ be a cumulative density function (*cdf*) of a given continuous random variable. It is possible that $G(x)$ depends on one real parameter $\xi \in \mathbb{R}$ or a parameter vector $\xi \in \mathbb{R}^d$, where $d \geqslant 2$. But for the sake of notation, we will ignore $\xi$ in the notation. According to the literature (see for example Rahman et al. (2020b), Ali and Athar (2021), Imliyangba et al. (2021)), two methods have been used to obtain a new *cdf* $F(x)$ called "cubic transmutation" of $G(x)$:

- the first one is based on the use of order statistics (Granzotto et al. (2017), Rahman et al. (2018a), Rahman et al. (2018b));
- the second one (AL-Kadim and Mohammed (2017), Rahman et al. (2019b), Rahman et al. (2023)) is based on the use of the following formula which is a special case of a more general formula proposed by Alzaatreh et al. (2013):

$$F(x) = R(G(x)) = \int_0^{G(x)} r(t)\, dt, \tag{3}$$

  where $R(t)$ is a polynomial *cdf* of degree 3 increasing, right-continuous such that $R(0) = 0$ and $R(1) = 1$ and $r(t)$ is the polynomial probability density function (*pdf*) of degree 2 linked to $R(t)$.

Note that if $g(x)$ is the *pdf* linked to $G(x)$, then the *pdf* linked to the *cdf* $F(x)$ is

$$f(x) = F'(x) = g(x)\, r(G(x)). \tag{4}$$

In the remainder of this section, we review six cubic transmutation formulas based on one or the other of the two methods.

### *2.1. First formula: Granzotto et al. (2017)*

According to Saha et al. (2024), the first cubic transmutation formula is due to Granzotto et al. (2017). The latter authors proved that the quadratic transmutation formula (1) of Shaw and Buckley (2009) could be rewritten as the *cdf* of a mixture of order statistics. Indeed, if $X_1$ and $X_2$ are two independent random variables sampled from the probability distribution with *cdf* $G(x)$ and if $X_{1:2}$, $X_{2:2}$ represent the corresponding order statistics, then the mixture random variable

$$X = \begin{cases} X_{1:2} & \text{with probability } \pi, \\ X_{2:2} & \text{with probability } 1-\pi, \end{cases} \tag{5}$$

where $0 \leqslant \pi \leqslant 1$, has the *cdf*

$$\begin{aligned} F_S(x) &= \pi \mathbb{P}(X_{1:2} \leqslant x) + (1-\pi)\mathbb{P}(X_{2:2} \leqslant x) \\ &= \pi \left[1-(1-G(x))^2\right] + (1-\pi)G^2(x) \\ &= 2\pi G(x) + (1-2\pi)G^2(x) \\ &= (1+\lambda)G(x) - \lambda G^2(x), \end{aligned}$$

where $\lambda = 2\pi - 1$ is such that $-1 \leqslant \lambda \leqslant 1$. They extended this reasoning to three variables to obtain their cubic transmutation formula. Indeed, if $X_1$, $X_2$ and $X_3$ are three independent random variables sampled from the probability distribution with *cdf* $G(x)$ and if $X_{1:3}$, $X_{2:3}$, $X_{3:3}$ represent the corresponding order statistics, then the mixture random variable

$$X = \begin{cases} X_{1:3} & \text{with probability } \pi_1, \\ X_{2:3} & \text{with probability } \pi_2, \\ X_{3:3} & \text{with probability } \pi_3, \end{cases} \tag{6}$$

where for all $i = 1, 2, 3$, $0 \leqslant \pi_i \leqslant 1$ and $\pi_1 + \pi_2 + \pi_3 = 1$, has the *cdf*

$$F_G(x) = \pi_1 \mathbb{P}(X_{1:3} \leqslant x) + \pi_2 \mathbb{P}(X_{2:3} \leqslant x) + \pi_3 \mathbb{P}(X_{3:3} \leqslant x),$$

where for all $i = 1, 2, 3$, according to the formula of the *cdf* of order statistics (see for example, David and Mishriky (1968)),

$$\mathbb{P}(X_{i:3} \leqslant x) = \sum_{r=i}^{3} C_3^r G^r(x)(1-G(x))^{3-r}. \tag{7}$$

So, Granzotto et al. (2017) obtained

$$F_G(x) = 3\pi_1 G(x) + 3(\pi_2 - \pi_1)G^2(x) + (1 - 3\pi_2)G^3(x) \tag{8}$$

which may be rewritten

$$F_G(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1)G^2(x) + (1-\lambda_2)G^3(x), \tag{9}$$

where $\lambda_1 = 3\pi_1$ and $\lambda_2 = 3\pi_2$. Notice that setting $\lambda_1 = \lambda_2 = 1$ enables to get the baseline distribution with *cdf* $G(x)$. The range proposed by Granzotto et al. (2017) for parameters $\lambda_1$ and $\lambda_2$ is

$$\mathcal{S}_G = \Big\{(\lambda_1, \lambda_2) : 0 \leqslant \lambda_1 \leqslant 1 \text{ and } -1 \leqslant \lambda_2 \leqslant 1\Big\}. \tag{10}$$

The corresponding *pdf* is

$$f_G(x) = g(x)\left[\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1-\lambda_2)G^2(x)\right]. \tag{11}$$

The cubic transmutation of Granzotto et al. (2017) will be denoted $CT_G$.

**Remark 1.** It is worth mentioning that the $CT_G$ is not a generalization of quadratic transmutation. Indeed, the formula (1) can be obtained from the formula (9) only by setting $\lambda_1 = 1+\lambda$ and $\lambda_2 = 1$. By doing so, we would have $0 \leqslant 1+\lambda \leqslant 1$ or, equivalently, $-1 \leqslant \lambda \leqslant 0$ which is different from the parameter range of quadratic transmutation.

**Remark 2.** Aslam et al. (2018) extended the formula (9) by replacing $G(x)$ by $G(x)^\alpha$, where $\alpha > 0$. They obtained a new family which they also called transmuted family of distributions. However this can be considered as a compounding of transmutation and exponentiation (see Ali et al. (2007)) so it will not be considered in this study. Riffi (2019) also proposed new families of distributions called higher rank transmutation (*HRT*) maps to generate more flexible and tractable lifetime families of probability models. Merovci et al. (2016) introduced a new family called another generalized transmuted (*AGT*) family of distributions by replacing $G(x)$ by $1-(1-G(x))^\alpha$ in (1). Nofal et al. (2017) proposed the generalized transmuted (*GT*) family of distributions by replacing $G(x)$ by $G^a(x)$ and $G^2(x)$ by $G^{a+b}(x)$ in (1), where $a$ and $b$ are positive parameters. But as for Aslam et al. (2018), *AGT*, *HRT* and *GT* can be considered as compounding of transmutation and exponentiation so they do not fit with the scope of this study.

### *2.2. Second formula: AL-Kadim and Mohammed (2017)*

The cubic transmutation of AL-Kadim and Mohammed (2017) (that we will denote $CT_A$) corresponds to the definition of the *cdf*

$$R(t) = (1+\lambda)t - 2\lambda t^2 + \lambda t^3$$

for $t \in [0,1]$ and the use of Formula (3). For a baseline *cdf* $G(x)$, it corresponds to the *cdf*

$$F_A(x) = (1+\lambda)G(x) - 2\lambda G^2(x) + \lambda G^3(x), \tag{12}$$

where $-1 \leqslant \lambda \leqslant 1$. The corresponding *pdf* is

$$f_A(x) = g(x)\left[(1+\lambda) - 4\lambda G(x) + 3\lambda G^2(x)\right]. \tag{13}$$

Setting $\lambda = 0$ enables to get the baseline distribution. It is important to note that this form of cubic transmutation is not a generalization of quadratic transmutation because the term $\lambda G^3(x)$ can only vanish if $\lambda = 0$, which would also force the term $2\lambda G^2(x)$ to vanish.

### 2.3. *Third formula: Rahman et al. (2018a)*

By permuting the probabilities $\pi_1$ and $\pi_3$ in Equation (8) and taking into account the relation $\pi_1+\pi_2+\pi_3 = 1$, Rahman et al. (2018a) obtained another form of cubic transmutation whose *cdf* is defined by

$$F(x) = (3-3\pi_1-3\pi_2)G(x) + (3\pi_1+6\pi_2-3)G^2(x) - (3\pi_2-1)G^3(x). \tag{14}$$

By setting $\lambda_1 = 2-3\pi_1-3\pi_2$ and $\lambda_2 = 3\pi_2-1$, they obtained

$$F_{R18a}(x) = (1+\lambda_1)G(x) + (\lambda_2-\lambda_1)G^2(x) - \lambda_2 G^3(x), \tag{15}$$

where the couple of new parameters $(\lambda_1, \lambda_2)$ belongs to the set

$$\mathcal{S}_{R18a} = \Big\{(\lambda_1,\lambda_2) : -1 \leqslant \lambda_1 \leqslant 1,\ \ -1 \leqslant \lambda_2 \leqslant 1 \ \text{ and } \ -2 \leqslant \lambda_1+\lambda_2 \leqslant 1\Big\}. \tag{16}$$

The corresponding *pdf* is

$$f_{R18a}(x) = g(x)\left[(1+\lambda_1) + 2(\lambda_2-\lambda_1)G(x) - 3\lambda_2 G^2(x)\right]. \tag{17}$$

One gets the baseline distribution by setting $\lambda_1 = \lambda_2 = 0$ and the quadratic transmutation by setting $\lambda_1 = \lambda$ and $\lambda_2 = 0$.

### 2.4. *Fourth formula: Rahman et al. (2018b)*

Considering Equation (14) and setting $\lambda_1 = 1-3\pi_1$ and $\lambda_2 = 1-3\pi_2$, Rahman et al. (2018b) obtained a new *cdf* in the form

$$F_{R18b}(x) = (1+\lambda_1+\lambda_2)G(x) - (\lambda_1+2\lambda_2)G^2(x) + \lambda_2 G^3(x), \tag{18}$$

and the corresponding *pdf* is

$$f_{R18b}(x) = g(x)\left[(1+\lambda_1+\lambda_2) - 2(\lambda_1+2\lambda_2)G(x) + 3\lambda_2 G^2(x)\right]. \tag{19}$$

One gets the baseline distribution by setting $\lambda_1 = \lambda_2 = 0$ and the quadratic transmutation by setting $\lambda_1 = \lambda$ and $\lambda_2 = 0$.

The range of the parameters are

$$\mathcal{S}_{R18b} = \Big\{(\lambda_1,\lambda_2) : -1 \leqslant \lambda_1 \leqslant 1 \ \text{ and } \ 0 \leqslant \lambda_2 \leqslant 1\Big\}. \tag{20}$$

### *2.5. Fifth formula: Rahman et al. (2019b)*

Using the formula (3) for the *pdf*

$$r(t) = (1-\lambda) + 6\lambda t - 6\lambda t^2$$

on $[0,1]$, Rahman et al. (2019b) defined another form of cubic transmutation with the following *cdf*:

$$F_{R19}(x) = (1-\lambda)G(x) + 3\lambda G^2(x) - 2\lambda G^3(x), \tag{21}$$

where $-1 \leqslant \lambda \leqslant 1$. The corresponding *pdf* is

$$f_{R19}(x) = g(x)\left[(1-\lambda) + 6\lambda G(x) - 6\lambda G^2(x)\right]. \tag{22}$$

This cubic transmutation formula provides the baseline distribution for $\lambda = 0$. For the same reasons as those mentioned for the formula of AL-Kadim and Mohammed (2017), this form of cubic transmutation is not a generalization of quadratic transmutation.

**Remark 3.** The transmutation formula (21) has also been considered by Aslam et al. (2020) and called modified transform (*MT*) distribution. The latter studied the statistical characteristics of *MT* distributions and applied it using Weibull distribution as baseline distribution. Application on two real datasets suggest that their proposed *MT* Weibull distribution fits the data better than the *CT* Weibull using the formula of Granzotto et al. (2017).

### *2.6. Sixth formula: Rahman et al. (2023)*

Using the formula (3) for the *pdf*

$$r(t) = 1 - \lambda(\eta-1) + 2\lambda(2\eta-1)t - 3\lambda\eta t^2$$

on $[0,1]$, Rahman et al. (2023) have developed a modified cubic transmuted family of distributions. The *cdf* of this family of distributions is defined for all $x \in \mathbb{R}$ by

$$F_{R23}(x) = \left[1-\lambda(\eta-1)\right]G(x) + \lambda(2\eta-1)G^2(x) - \lambda\eta G^3(x), \tag{23}$$

where

$$(\lambda,\eta) \in \mathcal{S}_{R23} = \Big\{(\lambda,\eta) : -1 \leqslant \lambda \leqslant 1 \text{ and } 0 \leqslant \eta \leqslant 2\Big\} \tag{24}$$

and the corresponding *pdf* is

$$f_{R23}(x) = g(x)\left[1-\lambda(\eta-1) + 2\lambda(2\eta-1)G(x) - 3\lambda\eta G^2(x)\right]. \tag{25}$$

When $\lambda = 0$, one gets the baseline distribution. Setting $\eta = 0$ enables to get the quadratic transmutation of Shaw and Buckley (2009) and setting $\eta = 2$ enables to get the cubic transmutation of Rahman et al. (2019b).

### 3. Main theoretical comparison results

For the six cubic transmutation models presented in Section 2, the constraints on the new parameters are very important to ensure that the new *cdf* verify all the properties required for a cumulative distribution function. For each model, the parameter range (or the set of all possible values of the parameters) are as important as the definition set would be for a function. In this sense, the comparison of two transmutation formulas must be done not only in terms of expression of *cdf* but also in terms of range of parameters. In this section, we give some theoretical comparison results on these different formulas. We carry out a critical study of different formulas, in particular by re-examining the parameter ranges in such a way as to always guarantee that the new distribution is well defined (i.e. is truly a probability distribution). As suggested by some comments and calculations in Hameldarbandi and Yilmaz (2020) and Yilmaz (2025), the theoretical study of the range of parameters is far from easy when the cubic transmutation contains two additional parameters and it is possible to then resort to a graphical study. The study made in this section will lead us to propose modified versions of the parameter range allowing us to extend the distributions obtained. We will also study the formulas to determine if some of the studied families of distributions can be obtained as subfamilies of the others.

Our main theoretical comparison results are given by Proposition 1 to Proposition 5 hereafter.

**Proposition 1.** *Let $G(x)$ be a cdf of a given random variable. The Granzotto Cubic Transmuted distribution of parameters $\lambda_1$ and $\lambda_2$ (which we will denote $CT_G(\lambda_1, \lambda_2)$) with cdf $F_G(x)$ (see Equation (9)) is not well defined under conditions (10). The $CT_G(\lambda_1, \lambda_2)$ distribution is rather well defined under the following modified conditions on $\lambda_1$ and $\lambda_2$:*

$$\mathcal{S}_{MG} = \Big\{(\lambda_1, \lambda_2) : 0 \leqslant \lambda_1 \leqslant 3, \;\; 0 \leqslant \lambda_2 \leqslant 3 \;\; \textit{and} \;\; 0 \leqslant \lambda_1 + \lambda_2 \leqslant 3\Big\}, \tag{26}$$

*and will be called modified Granzotto Cubic Transmutation and denoted $CT_{MG}$.*

*Proof.* By definition of the *cdf* of order statistics (see for example, David and Mishriky (1968)), the *cdf*

$$F_G(x) = 3\pi_1 G(x) + 3(\pi_2 - \pi_1)G^2(x) + (1 - 3\pi_2)G^3(x)$$

from Equation (8) is well defined under the conditions $0 \leqslant \pi_i \leqslant 1$ for all $i = 1, 2, 3$ and $\pi_1 + \pi_2 = 1 - \pi_3 \in [0, 1]$. Taking into account the relationships $\lambda_1 = 3\pi_1$ and $\lambda_2 = 3\pi_2$, the parameters $\lambda_1$ and $\lambda_2$ therefore satisfy the three constraints (26). Since the conditions (10) defined by Granzotto et al. (2017) are not included in the conditions (26), it is enough to find a counter-example to show that $CT_G$ is not well defined under conditions (10). Let us set $\lambda_1 = 0$ and $\lambda_2 = -0.5$. The couple $(\lambda_1, \lambda_2) = (0, -0.5)$ satisfies conditions (10) and the corresponding *cdf* is

$$F_G(x) = -0.5\,G^2(x) + 1.5\,G^3(x) = 1.5\,G^2(x)\left[G(x) - \frac{1}{3}\right].$$

Let $q_{1/3}$ be the quantile of order $1/3$ of the baseline distribution associated with the *cdf* $G(x)$. We have

$$\begin{cases} 1.5\, G^2(x)\left[G(x)-\frac{1}{3}\right] \leqslant 0 & \text{if} \quad x \leqslant q_{1/3}, \\ 1.5\, G^2(x)\left[G(x)-\frac{1}{3}\right] \geqslant 0 & \text{if} \quad x \geqslant q_{1/3}, \end{cases}$$

which proves that $F_G(x)$ is not a cumulative distribution function for $\lambda_1 = 0$ et $\lambda_2 = -0.5$.

**Remark 4.** By relaxing the assumption of mixture density of order statistics which requires that $\lambda_1 \geqslant 0$ and $\lambda_2 \geqslant 0$ and by using the method based on Formula (3), the fact that $f_G(x)$ is a well defined *pdf* is equivalent to the fact that the function

$$r(t) = \lambda_1 + 2(\lambda_2 - \lambda_1)t + 3(1-\lambda_2)t^2 \tag{27}$$

is positive and thus a *pdf* on $[0,1]$. Figure 1 presents a graphical study (for $\lambda_1 \in [-0.5, 4.5]$ and $\lambda_2 \in [-3.5, 3.5]$ with discretization step equal to 0.1 for both parameters) of whether or not, $r(t)$ defined by Equation (27) is a positive function on $[0,1]$. In this figure, the range (10) proposed by Granzotto et al. (2017) is represented by a square area delimited in red dotted lines while the modified range (26) proposed in this paper is represented by the black triangular area. We notice that there exist values of $\lambda_1$ and $\lambda_2$ outside the conditions (26) for which $r(t)$ is positive but the shape of the graph does not suggest any simple expression of the constraints on these parameters apart from the conditions (26).

**Remark 5.** The fact that the Granzotto Cubic Transmuted distribution is not well defined has also been pointed out by Hameldarbandi and Yilmaz (2020). However, the proof and the modified range given in our paper are different from the ones given by these latter authors. Indeed, the approach used by Hameldarbandi and Yilmaz (2020) is based on a reparametrization leading to the cubic transmutation of Rahman et al. (2018a) while the approach used in the present paper is more direct and more simple.

**Remark 6.** Since the Granzotto cubic transmuted distribution is not well defined, it does not make sense to compare other *CT* distributions with it but rather with its modified version.

**Proposition 2.** *Let $G(x)$ be the cdf of a random variable.*

*(i) For all $\lambda$ such that $-1 \leqslant \lambda \leqslant 1$, the $CT_A(\lambda)$ family linked to $G(x)$ is a sub-family of the $CT_{R18a}(\lambda_1, \lambda_2)$ family of distributions linked to $G(x)$.*

*(ii) The $CT_A(\lambda)$ family of distributions defined by Equation (12) is not a sub-family of the $CT_G(\lambda_1, \lambda_2)$ family of distributions if $\lambda \neq 0$.*

*(iii) The $CT_A(\lambda)$ family of distributions is rather a sub-family of the $CT_{MG}(\lambda_1, \lambda_2)$ family of distributions for $-1 \leqslant \lambda \leqslant 1$.*

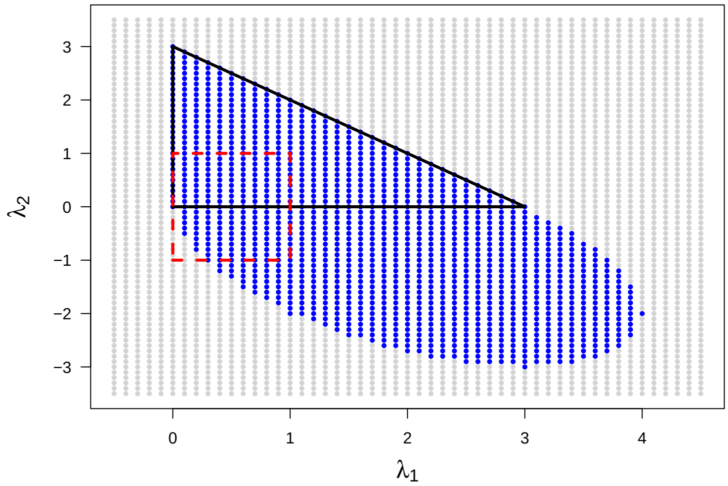


**Fig. 1.** Graphical study for different values of $\lambda_1$ and $\lambda_2$ of whether or not $r(t) = \lambda_1 + 2(\lambda_2 - \lambda_1)t + 3(1 - \lambda_2)t^2$ is positive on $[0,1]$ (blue colour means "yes" and gray means "no").

*(iv) Moreover, the $CT_A(\lambda)$ distribution linked to $G(x)$ is well defined under the extended condition $-1 \leqslant \lambda \leqslant 3$ and will be called modified AL-Kadim Cubic Transmutation and denoted $CT_{MA}(\lambda)$.*

**Proof**. Let us proceed by case.

(i) Let $\lambda \in [-1,1]$. Equation (13) can be rewritten under the form of Equation (17) by setting $\lambda_1 = \lambda$ and $\lambda_2 = -\lambda$. We then have $\lambda_1 = \lambda \in [-1,1]$, $\lambda_2 = -\lambda \in [-1,1]$ and $\lambda_1 + \lambda_2 = \lambda - \lambda = 0 \in [-2,1]$ so we can conclude that the $CT_A(\lambda)$ family is a sub-family of the $CT_{R18a}(\lambda_1, \lambda_2)$ family of distributions.

(ii) Let us first prove by absurd that the $CT_A(\lambda)$ family of distributions is not a sub-family of the $CT_G(\lambda_1, \lambda_2)$ family if $\lambda \neq 0$. Certainly, as noted by Saha et al. (2024), Equation (13) can be rewritten under the form of Equation (11) by setting $\lambda_1 = 1+\lambda$ and $\lambda_2 = 1 - \lambda$. However, there is a small subtlety that makes all the difference. It is that under such a change of variables, we would have

$$
\begin{aligned}
(0 \leqslant \lambda_1 \leqslant 1, \quad -1 \leqslant \lambda_2 \leqslant 1) &\iff (0 \leqslant 1+\lambda \leqslant 1, \quad -1 \leqslant 1-\lambda \leqslant 1) \\
&\iff (-1 \leqslant \lambda \leqslant 0, \quad 0 \leqslant \lambda \leqslant 2) \\
&\iff (\lambda = 0).
\end{aligned}
$$

Thus, the $CT_A(\lambda)$ family of distributions is a sub-family of the $CT_G(\lambda_1, \lambda_2)$ family only if $\lambda = 0$.

(iii) Let us now prove that the $CT_A(\lambda)$ family of distributions is a sub-family of the $CT_{MG}$ family of distributions for all $\lambda \in [-1, 1]$. Let $\lambda_1 = 1 + \lambda$ and $\lambda_2 = 1 - \lambda$.

$$\begin{aligned}
(0 \leqslant \lambda_1 \leqslant 3, \quad 0 \leqslant \lambda_2 \leqslant 3) &\iff (0 \leqslant 1 + \lambda \leqslant 3, \quad 0 \leqslant 1 - \lambda \leqslant 3) \\
&\iff (-1 \leqslant \lambda \leqslant 2, \quad -2 \leqslant \lambda \leqslant 1) \\
&\iff (-1 \leqslant \lambda \leqslant 1)
\end{aligned}$$

(iv) Let us now prove that the $CT_A(\lambda)$ distribution is well defined under the extended condition $-1 \leqslant \lambda \leqslant 3$. The *pdf* (13) is given by $f_A(x) = g(x)\, r(G(x))$, where for all $t \in [0, 1]$, $r(t) = (1 + \lambda) - 4\lambda t + 3\lambda t^2$ and, according to Alzaatreh et al. (2013), $f_A(x)$ is well defined if $r(t)$ is a *pdf* on $[0, 1]$. We have $\int_0^1 r(t)\, dt = 1$. So, for $r(t)$ to be a *pdf*, it is enough to check that for all $t \in [0, 1]$, $r(t) \geqslant 0$. We have $r'(t) = 2\lambda(3t - 2)$. A simple analysis of the sign of $r(t)$ gives:

(a) For $\lambda \leqslant 0$, $r(t)$ increases from $1 + \lambda$ to $(1 - \lambda/3)$ on $[0, 2/3]$ and next decreases to 1 on $[2/3, 1]$. So, it is enough that $\lambda \geqslant -1$.

(b) For$\lambda \geqslant 0$, $r(t)$ decreases and next increases in the same intervals to the same values. Thus, it is enough that $\lambda \leqslant 3$.

Thus, $r(t)$ is positive on $[0, 1]$ if $-1 \leqslant \lambda \leqslant 0$ or $0 \leqslant \lambda \leqslant 3$. We conclude that $r(t)$ is positive on $[0, 1]$ if $-1 \leqslant \lambda \leqslant 3$. ■

**Proposition 3.** *Let $G(x)$ be the cdf of a random variable.*

*(i) The cubic transmuted family of distributions of parameters $\lambda_1$ and $\lambda_2$ proposed by Rahman et al. (2018a) (denoted $CT_{R18a}(\lambda_1, \lambda_2)$) linked to $G(x)$ is well defined under conditions (16).*

*(ii) The $CT_{R18a}(\lambda_1, \lambda_2)$ family is also well defined under the extended range*

$$\mathcal{S}_{MR18a} = \Big\{(\lambda_1, \lambda_2) : -1 \leqslant \lambda_1 \leqslant 2, \quad -1 \leqslant \lambda_2 \leqslant 2 \;\; \textit{and} \;\; -2 \leqslant \lambda_1 + \lambda_2 \leqslant 1\Big\} \tag{28}$$

*and will be called modified $CT_{R18a}$ and denoted $CT_{MR18a}$.*

*(iii) For all $(\lambda_1, \lambda_2) \in \mathcal{S}_{MR18a}$, the $CT_{MR18a}(\lambda_1, \lambda_2)$ family of distributions linked to $G(x)$ is equivalent to the $CT_{MG}(\lambda_1 + 1, \lambda_2 + 1)$ family of distributions linked to $G(x)$.*

*(iv) For all $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$, the $CT_{MG}(\lambda_1, \lambda_2)$ family of distributions linked to $G(x)$ is equivalent to the $CT_{MR18a}(\lambda_1 - 1, \lambda_2 - 1)$ family of distributions linked to $G(x)$.*

**Proof**. We start with the proof of (ii).

*Proof of (ii)*. By definition of the *cdf* of order statistics, the *cdf*

$$F(x) = (3 - 3\pi_1 - 3\pi_2)G(x) + (3\pi_1 + 6\pi_2 - 3)G^2(x) - (3\pi_2 - 1)G^3(x)$$

from Equation (8) is well defined under the conditions $0 \leqslant \pi_i \leqslant 1$ for all $i = 1, 2, 3$ and $\pi_1 + \pi_2 = 1 - \pi_3 \in [0, 1]$. Since $\lambda_1 = 2 - 3\pi_1 - 3\pi_2$ and $\lambda_2 = 3\pi_2 - 1$, we can write $\lambda_1 = 2 - 3(1 - \pi_3) = 3\pi_3 - 1$, $\lambda_2 = 3\pi_2 - 1$ and $\lambda_1 + \lambda_2 = 1 - 3\pi_1$. Taking into account the range of $\pi_1$, $\pi_2$ and $\pi_3$, we deduce the three constraints (28).

*Proof of (i)*. It is immediate from ii) since the conditions (16) set by Rahman et al. (2018a) on $\lambda_1$ and $\lambda_2$ are included in conditions (28).

*Proof of (iii)*. Let $(\lambda_1, \lambda_2) \in \mathcal{S}_{MR18a}$. It is easily seen that by replacing $\lambda_1$ by $\lambda_1' = 1 + \lambda_1$ and $\lambda_2$ by $\lambda_2' = 1 + \lambda_2$ in (9), we obtain (15). We also easily verify that $(\lambda_1, \lambda_2) \in \mathcal{S}_{MR18a}$ if and only if $(1 + \lambda_1, 1 + \lambda_2) \in \mathcal{S}_{MG}$. Indeed,

$$\begin{aligned}
(\lambda_1, \lambda_2) \in \mathcal{S}_{MR18a} &\iff \begin{cases} -1 \leqslant \lambda_1 \leqslant 2 \\ -1 \leqslant \lambda_2 \leqslant 2 \\ -2 \leqslant \lambda_1 + \lambda_2 \leqslant 1 \end{cases} \\
&\iff \begin{cases} 0 \leqslant \lambda_1 + 1 \leqslant 3 \\ 0 \leqslant \lambda_2 + 1 \leqslant 3 \\ 0 \leqslant \lambda_1 + \lambda_2 + 2 \leqslant 3 \end{cases} \\
&\iff (\lambda_1 + 1, \lambda_2 + 1) \in \mathcal{S}_{MG}
\end{aligned}$$

which completes the proof because $(\lambda_1 + 1) + (\lambda_2 + 1) = \lambda_1 + \lambda_2 + 2$.

*Proof of (iv)*. Let $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$. It is easily seen that by replacing $\lambda_1$ by $\lambda_1' = \lambda_1 - 1$ and $\lambda_2$ by $\lambda_2' = \lambda_2 - 1$ in (15), we obtain (9). We also easily verify that $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$ if and only if $(\lambda_1 - 1, \lambda_2 - 1) \in \mathcal{S}_{MR18a}$. Indeed,

$$\begin{aligned}
(\lambda_1, \lambda_2) \in \mathcal{S}_{MG} &\iff \begin{cases} 0 \leqslant \lambda_1 \leqslant 3 \\ 0 \leqslant \lambda_2 \leqslant 3 \\ 0 \leqslant \lambda_1 + \lambda_2 \leqslant 3 \end{cases} \\
&\iff \begin{cases} -1 \leqslant \lambda_1 - 1 \leqslant 2 \\ -1 \leqslant \lambda_2 - 1 \leqslant 2 \\ -2 \leqslant \lambda_1 + \lambda_2 - 2 \leqslant 1 \end{cases} \\
&\iff (\lambda_1 - 1, \lambda_2 - 1) \in \mathcal{S}_{MR18a}
\end{aligned}$$

which completes the proof because $(\lambda_1 - 1) + (\lambda_2 - 1) = \lambda_1 + \lambda_2 - 2$. ■

**Remark**. By using the method based on Formula (3), the fact that $f_{R18a}(x)$ is a well defined *pdf* is equivalent to the fact that the function

$$r(t) = (1+\lambda_1) + 2(\lambda_2 - \lambda_1)t - 3\lambda_2 t^2 \tag{29}$$

is positive and thus a *pdf* on $[0,1]$. Figure 2 presents a graphical study (for $\lambda_1 \in [-1.5, 3.5]$ and $\lambda_2 \in [-4.5, 2.5]$ with discretization step equal to 0.1 for both parameters) of whether or not, $r(t)$ defined by Equation (29) is a positive function on $[0,1]$. In this figure, the range (16) proposed by Rahman et al. (2018a) is represented by a red dotted triangular area while the modified range (28) proposed in this paper is represented by the black triangular area. We also notice that there exist values of $\lambda_1$ and $\lambda_2$ outside the conditions (28) for which $r(t)$ is positive but the shape of the graph does not suggest any simple expression of the constraints on these parameters apart from the conditions (28).

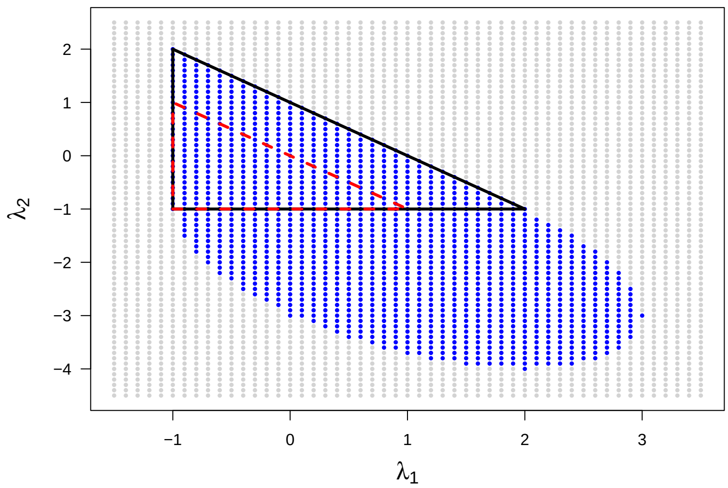


**Fig. 2.** Graphical study for different values of $\lambda_1$ and $\lambda_2$ of whether or not $r(t) = (1+\lambda_1) + 2(\lambda_2 - \lambda_1)t - 3\lambda_2 t^2$ is positive on $[0,1]$ (blue colour means "yes" and gray means "no").

**Proposition 4.** *Let $G(x)$ be the cdf of a random variable.*

*(i) The cubic transmuted family of distributions of parameters $\lambda_1$ and $\lambda_2$ proposed by Rahman et al. (2018b) (denoted $CT_{R18b}(\lambda_1, \lambda_2)$) linked to $G(x)$ is well defined under conditions (20).*

*(ii) The $CT_{R18b}(\lambda_1, \lambda_2)$ family is also well defined under the extended range*

$$\mathcal{S}_{MR18b} = \Big\{(\lambda_1, \lambda_2) : -2 \leqslant \lambda_1 \leqslant 1,\ -2 \leqslant \lambda_2 \leqslant 1 \text{ and } -1 \leqslant \lambda_1 + \lambda_2 \leqslant 2\Big\} \quad (30)$$

*and will be called modified $CT_{R18b}$ and denoted $CT_{MR18b}$.*

*(iii) For all $(\lambda_1, \lambda_2) \in \mathcal{S}_{MR18b}$, the $CT_{MR18b}(\lambda_1, \lambda_2)$ family of distributions linked to $G(x)$ is equivalent to the $CT_{MG}(1+\lambda_1+\lambda_2, 1-\lambda_2)$ family of distributions linked to $G(x)$.*

*(iv) For all $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$, the $CT_{MG}(\lambda_1, \lambda_2)$ family of distributions linked to $G(x)$ is equivalent to the $CT_{MR18b}(\lambda_1 + \lambda_2 - 2, 1 - \lambda_2)$ family of distributions linked to $G(x)$.*

**Proof.**

*Proof of (ii).* By definition of the *cdf* of order statistics, the *cdf* from Equation (8) is well defined under the conditions $0 \leqslant \pi_i \leqslant 1$ for all $i = 1, 2, 3$ and $\pi_1 + \pi_2 = 1 - \pi_3 \in [0, 1]$. Since $\lambda_1 = 1 - 3\pi_1$ and $\lambda_2 = 1 - 3\pi_2$, we easily deduce that $-2 \leqslant \lambda_1 \leqslant 1$ and $-2 \leqslant \lambda_2 \leqslant 1$. We also have

$$\lambda_1 + \lambda_2 = 2 - 3(\pi_1 + \pi_2) = 2 - 3(1 - \pi_3) = 3\pi_3 - 1$$

hence $-1 \leqslant \lambda_1 + \lambda_2 \leqslant 2$.

*Proof of (i).* It is immediate from ii) since the conditions (20) set by Rahman et al. (2018b) on $\lambda_1$ and $\lambda_2$ are included in (30).

*Proof of (iii).* Let $(\lambda_1, \lambda_2) \in \mathcal{S}_{MR18b}$. It is easily seen that by replacing $\lambda_1$ by $\lambda_1' = 1 + \lambda_1 + \lambda_2$ and $\lambda_2$ by $\lambda_2' = 1 - \lambda_2$ in (9), we obtain (18). We also easily verify that $(\lambda_1, \lambda_2) \in \mathcal{S}_{MR18b}$ if and only if $(1 + \lambda_1 + \lambda_2, 1 - \lambda_2) \in \mathcal{S}_{MG}$. Indeed

$$\begin{aligned}
(\lambda_1, \lambda_2) \in \mathcal{S}_{MR18b} &\iff \begin{cases} -1 \leqslant \lambda_1 + \lambda_2 \leqslant 2 \\ \quad -2 \leqslant \lambda_2 \leqslant 1 \\ \quad -2 \leqslant \lambda_1 \leqslant 1 \end{cases} \\
&\iff \begin{cases} 0 \leqslant 1 + \lambda_1 + \lambda_2 \leqslant 3 \\ \quad 0 \leqslant 1 - \lambda_2 \leqslant 3 \\ \quad 0 \leqslant 2 + \lambda_1 \leqslant 3 \end{cases} \\
&\iff (1 + \lambda_1 + \lambda_2, 1 - \lambda_2) \in \mathcal{S}_{MG}
\end{aligned}$$

which completes the proof because $(1 + \lambda_1 + \lambda_2) + (1 - \lambda_2) = 2 + \lambda_1$.

*Proof of (iv).* Let $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$. By replacing $\lambda_1$ by $\lambda_1' = \lambda_1 + \lambda_2 - 2$ and $\lambda_2$ by $\lambda_2' = 1 - \lambda_2$ in (18), we obtain (9). We also easily verify that $(\lambda_1, \lambda_2) \in \mathcal{S}_{MG}$ if and only if $(\lambda_1 + \lambda_2 - 2, 1 - \lambda_2) \in \mathcal{S}_{MR18b}$. Indeed

$$
\begin{aligned}
(\lambda_1,\lambda_2)\in\mathcal{S}_{MG} &\iff \begin{cases} 0\leqslant \lambda_1+\lambda_2\leqslant 3\\ 0\leqslant\lambda_2\leqslant 3\\ 0\leqslant\lambda_1\leqslant 3\end{cases}\\
&\iff \begin{cases} -2\leqslant \lambda_1+\lambda_2-2\leqslant 1\\ -2\leqslant 1-\lambda_2\leqslant 1\\ -1\leqslant\lambda_1-1\leqslant 2\end{cases}\\
&\iff (\lambda_1+\lambda_2-2, 1-\lambda_2)\in\mathcal{S}_{MR18b}
\end{aligned}
$$

which completes the proof because $(\lambda_1+\lambda_2-2)+(1-\lambda_2)=\lambda_1-1$. ■

**Remark 7.** By using the method based on Formula (3), the fact that $f_{R18b}(x)$ is a well defined *pdf* is equivalent to the fact that the function

$$r(t) = (1+\lambda_1+\lambda_2) - 2(\lambda_1+2\lambda_2)t + 3\lambda_2 t^2 \tag{31}$$

is positive and thus a *pdf* on $[0,1]$. Figure 3 presents a graphical study (for $\lambda_1\in[-3.5,1.5]$ and $\lambda_2\in[-2.5,4.5]$ with discretization step equal to 0.1 for both parameters) of whether or not, $r(t)$ defined by Equation (31) is a positive function on $[0,1]$. In this figure, the range (20) proposed by Rahman et al. (2018b) is represented by a red dotted rectangular area while the modified range (30) proposed in this paper is represented by the black triangular area. We also notice that there exist values of $\lambda_1$ and $\lambda_2$ outside the conditions (30) for which $r(t)$ is positive but the shape of the graph does not suggest any simple expression of the constraints on these parameters apart from the conditions (30).

**Proposition 5.** *We have:*

*(i) The $CT_{R19}(\lambda)$ family of distributions defined by Equation (21) is not a sub-family of the $CT_G(\lambda_1,\lambda_2)$ family of distributions if $\lambda\neq 0$.*

*(ii) It is rather a sub-family of the $CT_{MG}(\lambda_1,\lambda_2)$ family of distributions for $-\frac{1}{2}\leqslant\lambda\leqslant 1$.*

*(iii) Moreover, the $CT_{R19}(\lambda)$ distribution is well defined under the extended condition $-2\leqslant\lambda\leqslant 1$ and will be called modified cubic Transmutation of Rahman et al. (2019b) and denoted $CT_{MR19}$.*

**Proof**. We proceed by case.

*Proof of (i).* Let us first prove by absurd that the $CT_{R19}(\lambda)$ family of distributions is not a sub-family of the $CT_G(\lambda_1,\lambda_2)$ family if $\lambda\neq 0$. Let $\lambda\in[-1,1]$. Equation (21) can be rewritten under the form of Equation (9) by setting $\lambda_1 = 1-\lambda$ and $\lambda_2 = 1+2\lambda$. We would have

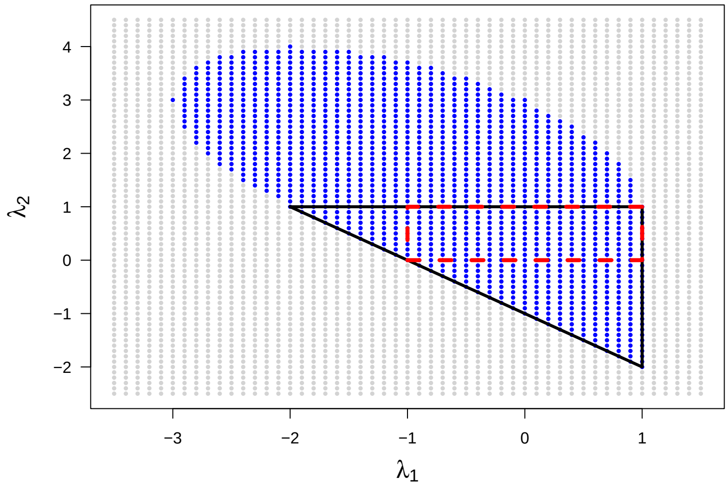


**Fig. 3.** Graphical study for different values of $\lambda_1$ and $\lambda_2$ of whether or not $r(t) = (1+\lambda_1+\lambda_2) - 2(\lambda_1+2\lambda_2)t + 3\lambda_2 t^2$ is positive on $[0,1]$ (blue colour means "yes" and gray means "no").

$$\begin{aligned}
(\lambda_1,\lambda_2) = (1-\lambda, 1+2\lambda) \in \mathcal{S}_G &\iff \begin{cases} 0 \leqslant 1-\lambda \leqslant 1 \\ -1 \leqslant 1+2\lambda \leqslant 1 \end{cases} \\
&\iff \begin{cases} 0 \leqslant \lambda \leqslant 1 \\ -1 \leqslant \lambda \leqslant 0 \end{cases} \\
&\iff \lambda = 0.
\end{aligned}$$

Thus, the $CT_{R19}(\lambda)$ family of distributions is a sub-family of the $CT_G(\lambda_1,\lambda_2)$ family only if $\lambda = 0$.

*Proof of (ii).* Let $\lambda \in [-1,1]$, $\lambda_1 = 1-\lambda$ and $\lambda_2 = 1+2\lambda$.

$$\begin{aligned}
\left.\begin{array}{r} 0 \leqslant \lambda_1 \leqslant 3 \\ 0 \leqslant \lambda_2 \leqslant 3 \\ 0 \leqslant \lambda_1+\lambda_2 \leqslant 3 \end{array}\right\} &\iff \begin{cases} 0 \leqslant 1-\lambda \leqslant 3 \\ 0 \leqslant 1+2\lambda \leqslant 3 \\ 0 \leqslant \lambda+2 \leqslant 3 \end{cases} \\
&\iff \begin{cases} -2 \leqslant \lambda \leqslant 1 \\ -\frac{1}{2} \leqslant \lambda \leqslant 1 \\ -2 \leqslant \lambda \leqslant 1 \end{cases} \\
&\iff -\frac{1}{2} \leqslant \lambda \leqslant 1.
\end{aligned}$$

*Proof of (iii).* Let us now prove that the $CT_{R19}$ distribution is well defined under the extended condition $-2 \leqslant \lambda \leqslant 1$. The *pdf* (22) is given by

$$f_{R19}(x) = g(x)\, r(G(x)),$$

where for all $t \in [0,1]$, $r(t) = (1-\lambda) + 6\lambda t - 6\lambda t^2$ and, according to Alzaatreh et al. (2013), $f_{R19}(x)$ is well defined if $r(t)$ is a *pdf* on $[0,1]$. We have

$$\int_0^1 r(t)\, dt = 1.$$

So for $r(t)$ to be a *pdf*, it is enough to check that for all $t \in [0,1]$, $r(t) \geqslant 0$. We have $r'(t) = -6\lambda(2t-1)$. Let us proceed by a simple analysis.

(a) For $\lambda \leqslant 0$, $r(t)$ decreases from $1-\lambda$ to $(\lambda/2)+1$ on $[0,1/2]$ and next increases to $1-\lambda$ on $[1/2,1]$. Thus, it is enough that $\frac{\lambda+2}{2} \geqslant 0$ i.e. $\lambda \geqslant -2$.

(b) For $\lambda \geqslant 0$, $r(t)$ increases and next decreases to the same values and on the same intervals. Thus, it is enough that $1-\lambda \geqslant 0$ i.e. $\lambda \leqslant 1$.

Finally, $r(t)$ is positive on $[0,1]$ if $-2 \leqslant \lambda \leqslant 0$ or $0 \leqslant \lambda \leqslant 1$. We conclude that $r(t)$ is positive on $[0,1]$ if $-2 \leqslant \lambda \leqslant 1$. ■

**Remark 8.** By using the method based on Formula (3), the fact that $f_{R23}(x)$ is a well defined *pdf* is equivalent to the fact that the function

$$r(t) = 1 - \lambda(\eta-1) + 2\lambda(2\eta-1)t - 3\lambda\eta t^2 \tag{32}$$

is positive and thus a *pdf* on $[0,1]$. As suggested by some comments and calculations in Hameldarbandi and Yilmaz (2020) and Yilmaz (2025), the theoretical determination of the range is far from easy when the cubic transmutation contains two parameters. We can then resort to a graphic study. Figure 4 presents a graphical study (for $\lambda \in [-3.2, 1.2]$ and $\eta \in [-3,3]$ with discretization step equal to 0.1 for both parameters) of whether or not $r(t)$ defined by Equation (32) is a positive function on $[0,1]$. In this figure, the range (24) proposed by Rahman et al. (2023) is represented by a rectangular area in red dashes. We also notice that there exist values of $\lambda$ and $\eta$ outside the conditions (24) for which $r(t)$ is positive but the shape of the graph does not suggest any simple expression of the constraints on these parameters apart from the conditions (24).

## 4. Illustration of the results using Pareto distribution

In this section, we illustrate our results using Pareto distribution as baseline distribution. We compare the Pareto distribution, transmuted Pareto (*TP*) distribution, the cubic transmuted Pareto (*CTP*) distributions and their modifications on real datasets using R software (R Core Team, 2024). For this purpose, we tested several datasets including those which have been used in

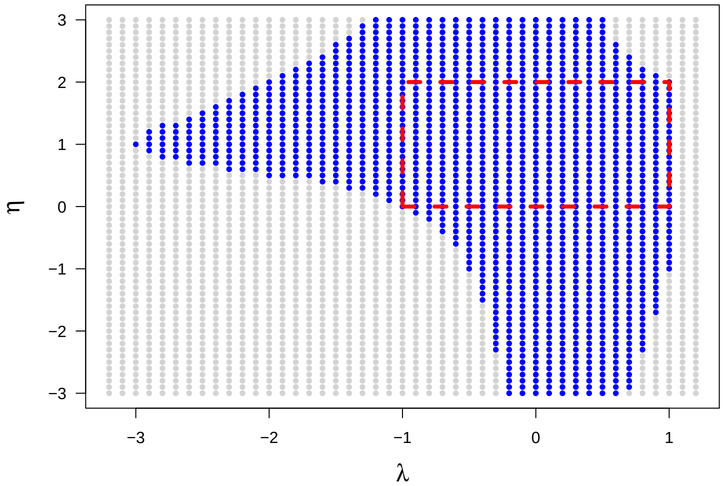


**Fig. 4.** Graphical study for different values of $\lambda$ and $\eta$ of whether or not $r(t) = (1+\lambda(1-\eta)) + 2\lambda(2\eta-1)t - 3\lambda\eta t^2$ is positive on $[0,1]$ (blue colour means "yes" and gray means "no").

the past for the quadratic and cubic transmutations of Pareto distribution in Merovci and Puka (2014), Ansari and Eledum (2018), Rahman et al. (2020a) and Rahman et al. (2021). The datasets selected and presented hereafter are mainly those which illustrate the theoretical results presented earlier in this paper.

For each dataset, the selection of the best model (the distribution which best fits each dataset) is done using the classical likelihood-based model selection criteria which are the negative of the log-likelihood ($-\log L^*$), the Akaike Information Criterion (*AIC*), the Akaike Information Criterion Corrected (*AICC*), the Bayesian Information Criterion (*BIC*) respectively defined as

$$AIC = -2\log L^* + 2k, \quad AICC = AIC + \frac{2k(k+1)}{n-(k+1)}, \quad BIC = -2\log L^* + k\log n,$$

where $k$ is the number of parameters to estimate, $n$ is the number of observations in the dataset and $\log L^*$ is the maximum value of the log-likelihood. For each dataset, the best model is the one with the smallest values for the criteria.

We first present the *pdfs*, likelihoods and maximum likelihood estimators (*MLE*) linked to each *CTP*.

*4.1. Probability density functions and MLE*

4.1.1. Probability density functions

Pareto distribution is defined by its *cdf*

$$G(x) = 1 - \left(\frac{x_0}{x}\right)^{\alpha}, \quad x \geqslant x_0, \tag{33}$$

and its *pdf*

$$g(x) = \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}}, \quad x \geqslant x_0, \tag{34}$$

where $x_0 \in \mathbb{R}_+^*$ and $\alpha \in \mathbb{R}_+^*$. Using the quadratic transmutation proposed by Shaw and Buckley (2009), Merovci and Puka (2014) developed the transmuted Pareto (*TP*) distribution.

Several authors have been interested in cubic transmuted Pareto distribution (*CTP*). Using the *CT* formula of AL-Kadim and Mohammed (2017), Ansari and Eledum (2018) proposed a *CTP* distribution that we will denote $CTP_A$ and whose *pdf* is

$$f_A(x) = \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}} \left[1 - 2\lambda \left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda \left(\frac{x_0}{x}\right)^{2\alpha}\right]. \tag{35}$$

Rahman et al. (2020a) have used the formula of Rahman et al. (2018a) and proposed a *CTP* distribution (that we will denote $CTP_{R18a}$) with corresponding *pdf*

$$f_{R18a}(x) = \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}} \left[(1 - \lambda_1 - \lambda_2) + 2(\lambda_1 + 2\lambda_2) \left(\frac{x_0}{x}\right)^{\alpha} - 3\lambda_2 \left(\frac{x_0}{x}\right)^{2\alpha}\right]. \tag{36}$$

Rahman et al. (2021) proposed another *CTP* distribution (that we will denote $CTP_{R18b}$) using the formula of Rahman et al. (2018b) and the corresponding *pdf* is

$$f_{R18b}(x) = \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}} \left[(1 - \lambda_1) + 2(\lambda_1 - \lambda_2) \left(\frac{x_0}{x}\right)^{\alpha} + 3\lambda_2 \left(\frac{x_0}{x}\right)^{2\alpha}\right]. \tag{37}$$

To the best of our knowledge, the other formulas of *CT* have not been applied so far to the Pareto distribution. The *CTP* linked to the formulas of Granzotto et al. (2017), Rahman et al. (2019b) and Rahman et al. (2023) will be respectively denoted $CTP_G$, $CTP_{R19}$ and $CTP_{R23}$ and their respective *pdfs* are

$$f_G(x) = \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}} \left[(3 - \lambda_1 - \lambda_2) + 2(\lambda_1 + 2\lambda_2 - 3) \left(\frac{x_0}{x}\right)^{\alpha} + 3(1 - \lambda_2) \left(\frac{x_0}{x}\right)^{2\alpha}\right], \tag{38}$$

$$f_{R19}(x) = \frac{\alpha x_0^{\alpha}}{x^{\alpha+1}} \left[(1 - \lambda) + 6\lambda \left(\frac{x_0}{x}\right)^{\alpha} - 6\lambda \left(\frac{x_0}{x}\right)^{2\alpha}\right] \tag{39}$$

and

$$f_{R23}(x) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}} \left[(1-\lambda) + 2\lambda(1+\eta)\left(\frac{x_0}{x}\right)^\alpha - 3\lambda\eta\left(\frac{x_0}{x}\right)^{2\alpha}\right]. \tag{40}$$

The modified versions of $CTP_A$, $CTP_{R18a}$, $CTP_{R18b}$, $CTP_G$ and $CTP_{R19}$ will be respectively denoted $CTP_{MA}$, $CTP_{MR18a}$, $CTP_{MR18b}$, $CTP_{MG}$ and $CTP_{MR19}$.

#### 4.1.2. Estimation of parameters using the maximum likelihood method

Let $x_1, \ldots, x_n$ be a random sample of size $n$ from a given *CTP* distribution. The general form of the log-likelihood is

$$\log L = \sum_{i=1}^{n} \log f(x_i), \tag{41}$$

where $f$ is the *pdf*.

Since for all $i = 1, \ldots, n$, $x_0 \leqslant x_i$, the maximum likelihood estimator (*MLE*) of $x_0$ is the minimum of the sample values i.e. $\min_{1\leqslant i\leqslant n} x_i$. The *MLEs* of the other parameters are obtained by maximizing the respective log-likelihoods of the different models defined below.

(1) The log-likelihood corresponding to both the $CTP_G$ and the $CTP_{MG}$ is

$$\begin{aligned}\log L_G(\alpha,\lambda_1,\lambda_2) = n\log(\alpha) + n\alpha\log(x_0) - (\alpha+1)\sum_{i=1}^{n}\log(x_i) \\ + \sum_{i=1}^{n}\log\Bigg[(3-\lambda_1-\lambda_2) + (2\lambda_1+4\lambda_2-6)\left(\frac{x_0}{x_i}\right)^\alpha \\ +(3-3\lambda_2)\left(\frac{x_0}{x_i}\right)^{2\alpha}\Bigg].\end{aligned} \tag{42}$$

The maximum likelihood estimates (*MLE*) $\hat{\alpha}$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of $\alpha$, $\lambda_1$ and $\lambda_2$ are such that

$$(\hat{\alpha}, \hat{\lambda}_1, \hat{\lambda}_2) = \underset{(\alpha,\lambda_1,\lambda_2)\in\mathbb{R}_+^*\times\mathcal{S}}{arg\ max}\Big(\log L_G(\alpha,\lambda_1,\lambda_2)\Big), \tag{43}$$

where $\mathcal{S} = \mathcal{S}_G$ if we consider the $CTP_G$ and $\mathcal{S} = \mathcal{S}_{MG}$ if we consider the $CTP_{MG}$.

(2) The log-likelihood corresponding to both the $CTP_A$ of Ansari and Eledum (2018) and the $CTP_{MA}$ is

$$\log L_A(\alpha,\lambda) = n\log(\alpha) + n\alpha\log(x_0) - (\alpha+1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left[1 - 2\lambda\left(\frac{x_0}{x_i}\right)^{\alpha} + 3\lambda\left(\frac{x_0}{x_i}\right)^{2\alpha}\right]. \quad (44)$$

The *MLE* $\hat{\alpha}$ and $\hat{\lambda}$ of $\alpha$ and $\lambda$ are such that

$$(\hat{\alpha},\hat{\lambda}) = \underset{(\alpha,\lambda)\in\mathbb{R}_+^*\times\mathcal{S}}{arg\ max}\ \log L_A(\alpha,\lambda), \quad (45)$$

where $\mathcal{S} = [-1,1]$ if we consider the $CTP_A$ and $\mathcal{S} = [-1,3]$ if we consider the $CTP_{MA}$.

(3) The log-likelihood corresponding to both $CTP_{R18a}$ Rahman et al. (2020a) and $CTP_{MR18a}$ is

$$\log L_{R18a}(\alpha,\lambda_1,\lambda_2) = n\log(\alpha) + n\alpha\log(x_0) - (\alpha+1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left[(1-\lambda_1-\lambda_2) + 2(\lambda_1+2\lambda_2)\left(\frac{x_0}{x_i}\right)^{\alpha} - 3\lambda_2\left(\frac{x_0}{x_i}\right)^{2\alpha}\right]. \quad (46)$$

The *MLE* $\hat{\alpha}$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of $\alpha$, $\lambda_1$ and $\lambda_2$ are such that

$$(\hat{\alpha},\hat{\lambda}_1,\hat{\lambda}_2) = \underset{(\alpha,\lambda_1,\lambda_2)\in\mathbb{R}_+^*\times\mathcal{S}}{arg\ max}\ \log L_{R18a}(\alpha,\lambda_1,\lambda_2), \quad (47)$$

where $\mathcal{S} = \mathcal{S}_{R18a}$ if we consider the $CTP_{R18a}$ and $\mathcal{S} = \mathcal{S}_{MR18a}$ if we consider the $CTP_{MR18a}$.

(4) The log-likelihood corresponding to both $CTP_{R18b}$ of Rahman et al. (2021) and $CTP_{MR18b}$ is

$$\log L_{R18b}(\alpha,\lambda_1,\lambda_2) = n\log(\alpha) + n\alpha\log(x_0) - (\alpha+1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left[(1-\lambda_1) + 2(\lambda_1-\lambda_2)\left(\frac{x_0}{x_i}\right)^{\alpha} + 3\lambda_2\left(\frac{x_0}{x_i}\right)^{2\alpha}\right]. \quad (48)$$

The *MLE* $\hat{\alpha}$, $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of $\alpha$, $\lambda_1$ and $\lambda_2$ are such that

$$(\hat{\alpha},\hat{\lambda}_1,\hat{\lambda}_2) = \underset{(\alpha,\lambda_1,\lambda_2)\in\mathbb{R}_+^*\times\mathcal{S}}{argmax}\ \log L_{R18b}(\alpha,\lambda_1,\lambda_2), \quad (49)$$

where $\mathcal{S} = \mathcal{S}_{R18b}$ if we consider the $CTP_{R18b}$ and $\mathcal{S} = \mathcal{S}_{MR18b}$ if we consider the $CTP_{MR18b}$.

(5) The log-likelihood corresponding to both $CTP_{R19}$ and $CTP_{MR19}$ is

$$\log L_{R19}(\alpha, \lambda) = n\log(\alpha) + n\alpha\log(x_0) - (\alpha+1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left[(1-\lambda) + 6\lambda\left(\frac{x_0}{x_i}\right)^{\alpha} - 6\lambda\left(\frac{x_0}{x_i}\right)^{2\alpha}\right]. \quad (50)$$

The *MLE* $\hat{\alpha}$ and $\hat{\lambda}$ of $\alpha$ and $\lambda$ are such that

$$(\hat{\alpha}, \hat{\lambda}) = \underset{(\alpha,\lambda)\in\mathbb{R}_+^*\times\mathcal{S}}{arg\ max}\ \log L_{R19}(\alpha, \lambda), \quad (51)$$

where $\mathcal{S} = [-1, 1]$ if we consider the $CTP_{R19}$ and $\mathcal{S} = [-2, 1]$ if we consider the $CTP_{MR19}$.

(6) The log-likelihood corresponding to $CTP_{R23}$ is

$$\log L_{R23}(\alpha, \lambda, \eta) = n\log(\alpha) + n\alpha\log(x_0) - (\alpha+1)\sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log\left[(1-\lambda) + 2\lambda(1+\eta)\left(\frac{x_0}{x_i}\right)^{\alpha} - 3\lambda\eta\left(\frac{x_0}{x_i}\right)^{2\alpha}\right]. \quad (52)$$

The *MLE* $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\eta}$ of $\alpha$, $\lambda$ and $\eta$ are such that

$$(\hat{\alpha}, \hat{\lambda}, \hat{\eta}) = \underset{(\alpha,\lambda,\eta)\in\mathbb{R}_+^*\times[-1,1]\times[0,2]}{arg\ max}\ \log L_{R23}(\alpha, \lambda, \eta). \quad (53)$$

All these constrained optimization problems are difficult to solve analytically and therefore require the use of a numerical optimization algorithm. One of the requirements for such algorithms is that they take inequality constraints into account. For our real data analysis hereafter, we will use the quasi-Newton BFGS algorithm (see Nocedal and Wright (2006)) through the R function **constrOptim**. To avoid problems related to the influence of the initial parameters necessary to start the algorithm (see for example, Geraldo (2021)), we tested a hundred initial values randomly chosen in the parameter space for each model and we retained the best initial value.

### *4.2. Floyd river dataset*

The Floyd data concerns the flood rates of the Floyd River (USA) from the years 1935 to 1973 (see Mudholkar and Hutson (1996) for more details on these data). Descriptive statistics for the Floyd data are summarized in Table 1.

**Table 1.** Descriptive statistics for the Floyd dataset

| Min | $Q_1$ | Median | Mean | $Q_3$ | Max |
|---|---|---|---|---|---|
| 318 | 1590 | 3570 | 6771 | 6725 | 71500 |

**Table 2.** Parameters estimated for unmodified models on the Floyd dataset

| Distributions | Estimations | | | |
|---|---|---|---|---|
| $CTP_G$ | $x_0 = 318$ | $\hat{\alpha} = 0.808$ | $\hat{\lambda}_1 = 0.104$ | $\hat{\lambda}_2 = -1$ |
| $CTP_A$ | $x_0 = 318$ | $\hat{\alpha} = 0.436$ | $\hat{\lambda} = -0.876$ | |
| $CTP_{R18a}$ | $x_0 = 318$ | $\hat{\alpha} = 0.718$ | $\hat{\lambda}_1 = -0.908$ | $\hat{\lambda}_2 = -1$ |
| $CTP_{R18b}$ | $x_0 = 318$ | $\hat{\alpha} = 0.602$ | $\hat{\lambda}_1 = -1$ | $\hat{\lambda}_2 = 0.138$ |
| $CTP_{R19}$ | $x_0 = 318$ | $\hat{\alpha} = 0.339$ | $\hat{\lambda} = 0.901$ | |
| $CTP_{R23}$ | $x_0 = 318$ | $\hat{\alpha} = 0.330$ | $\hat{\lambda} = 1$ | $\hat{\eta} = 1.845$ |
| *TP* | $x_0 = 318$ | $\hat{\alpha} = 0.586$ | $\hat{\lambda} = -0.91$ | |
| Pareto | $x_0 = 318$ | $\hat{\alpha} = 0.412$ | | |

**Table 3.** Criteria for comparing unmodified models on the Floyd dataset

| Distributions | $-\log L^*$ | *AIC* | *AICC* | *BIC* |
|---|---|---|---|---|
| $CTP_G$ | $375.626^{(1)}$ | $757.252^{(1)}$ | $757.938^{(1)}$ | $762.243^{(1)}$ |
| $CTP_{R18a}$ | $380.665^{(2)}$ | $767.330^{(2)}$ | $768.016^{(2)}$ | $772.321^{(2)}$ |
| $CTP_{R23}$ | $384.719^{(3)}$ | $775.438^{(5)}$ | $776.124^{(5)}$ | $780.429^{(5)}$ |
| $CTP_{R18b}$ | $384.915^{(4)}$ | $775.830^{(6)}$ | $776.516^{(6)}$ | $780.821^{(6)}$ |
| $CTP_{R19}$ | $385.161^{(5)}$ | $774.322^{(3)}$ | $774.655^{(3)}$ | $777.649^{(3)}$ |
| *TP* | $385.349^{(6)}$ | $774.698^{(4)}$ | $775.031^{(4)}$ | $778.025^{(4)}$ |
| $CTP_A$ | $387.052^{(7)}$ | $778.104^{(7)}$ | $778.437^{(7)}$ | $781.431^{(7)}$ |
| Pareto | $392.810^{(8)}$ | $787.620^{(8)}$ | $787.728^{(8)}$ | $789.284^{(8)}$ |

#### 4.2.1. Fitting with the initial (unmodified) distributions

Parameter estimates for the unmodified distributions are reported in Table 2 and Table 3 gives the different criteria for the unmodified models.

According to the results in Table 3, it seems that the distribution that best fits the Wheaton dataset is $CTP_G$. But in this case too, the $CTP_G$ distribution is not a true probability distribution in the rigorous sense of the term. Figure 5 represents the fitted *cdf* (left) and *pdf* (right) from unmodified Granzotto cubic transmutation ($CTP_G$) for Floyd data for $x \in [318, 850]$. We notice that the *cdf* is negative in the interval $[372.4243, 640.6569]$ and the *pdf* is negative in the interval $[341.1465, 505.9279]$.

#### 4.2.2. Fitting with modified distributions

Table 4 summarizes the estimated parameter values for the modified models, while Table 5 presents the model comparison criteria.

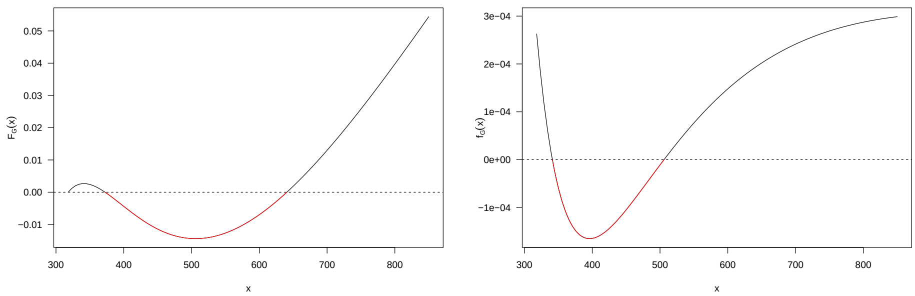


**Fig. 5.** The fitted *cdf* (left) and *pdf* (right) from unmodified Granzotto cubic transmutation ($CTP_G$) for Floyd data

**Table 4.** Estimated parameters for models modified on the Floyd dataset

| Distributions | Estimations | | | |
|---|---|---|---|---|
| $CTP_{MG}$ | $x_0 = 318$ | $\hat{\alpha} = 0.719$ | $\hat{\lambda}_1 = 0.092$ | $\hat{\lambda}_2 = 0$ |
| $CTP_{MA}$ | $x_0 = 318$ | $\hat{\alpha} = 0.436$ | $\hat{\lambda} = -0.876$ | |
| $CTP_{MR18a}$ | $x_0 = 318$ | $\hat{\alpha} = 0.719$ | $\hat{\lambda}_1 = -0.908$ | $\hat{\lambda}_2 = -1$ |
| $CTP_{MR18b}$ | $x_0 = 318$ | $\hat{\alpha} = 0.718$ | $\hat{\lambda}_1 = -1.908$ | $\hat{\lambda}_2 = 1$ |
| $CTP_{MR19}$ | $x_0 = 318$ | $\hat{\alpha} = 0.339$ | $\hat{\lambda} = 0.901$ | |
| $CTP_{R23}$ | $x_0 = 318$ | $\hat{\alpha} = 0.330$ | $\hat{\lambda} = 1$ | $\hat{\eta} = 1.845$ |
| *TP* | $x_0 = 318$ | $\hat{\alpha} = 0.586$ | $\hat{\lambda} = -0.91$ | |
| Pareto | $x_0 = 318$ | $\hat{\alpha} = 0.412$ | | |

**Table 5.** Criteria for modified models on the Floyd dataset (ranks in parentheses)

| Distributions | $-\log L^*$ | *AIC* | *AICC* | *BIC* |
|---|---|---|---|---|
| $CTP_{MG}$ | $380.665^{(1)}$ | $767.330^{(1)}$ | $768.016^{(1)}$ | $772.321^{(1)}$ |
| $CTP_{MR18a}$ | $380.665^{(1)}$ | $767.330^{(1)}$ | $768.016^{(1)}$ | $772.321^{(1)}$ |
| $CTP_{MR18b}$ | $380.665^{(1)}$ | $767.330^{(1)}$ | $768.016^{(1)}$ | $772.321^{(1)}$ |
| $CTP_{R23}$ | $384.719^{(4)}$ | $775.438^{(6)}$ | $776.124^{(6)}$ | $780.429^{(6)}$ |
| $CTP_{MR19}$ | $385.161^{(5)}$ | $774.322^{(4)}$ | $774.655^{(4)}$ | $777.649^{(4)}$ |
| *TP* | $385.349^{(6)}$ | $774.698^{(5)}$ | $775.031^{(5)}$ | $778.025^{(5)}$ |
| $CTP_{MA}$ | $387.052^{(7)}$ | $778.104^{(7)}$ | $778.437^{(7)}$ | $781.431^{(7)}$ |
| Pareto | $392.810^{(8)}$ | $787.620^{(8)}$ | $787.728^{(8)}$ | $789.284^{(8)}$ |

A comparison between Tables 2 and 4 shows that the parameters of the $CTP_G$ and $CTP_{R18b}$ distributions have changed, with the best-fitting distributions now being $CTP_{MG}$, $CTP_{MR18a}$ and $CTP_{MR18b}$. The modifications introduced have improved the $CTP_G$, ensuring it is well defined as a probability distribution and have elevated the $CTP_{R18b}$ from the fourth place to first (tied).

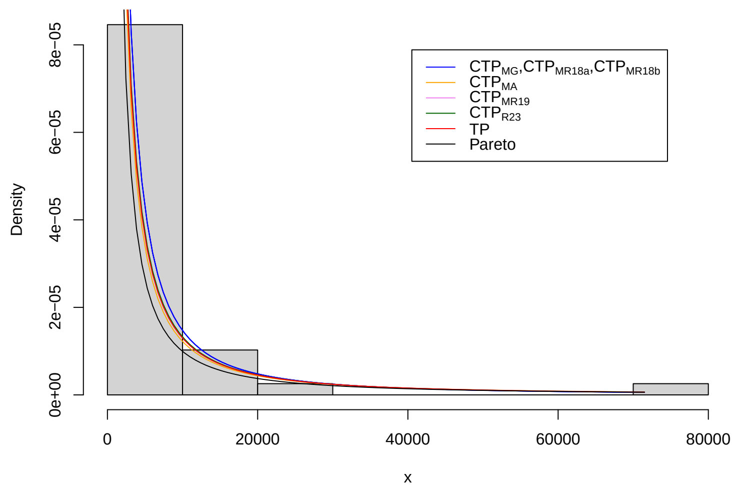


**Fig. 6.** Graphical comparison of fitting densities of modified models on the Floyd dataset

**Table 6.** Descriptive statistics for Population size data

| Min | $Q_1$ | Median | Mean | $Q_3$ | Max |
|---|---|---|---|---|---|
| 5.3E06 | 6.775E06 | 9.325E06 | 1.28E07 | 1.568E07 | 7.01E07 |

Figure 6 displays the histogram of the Floyd dataset along with the fitted densities of the modified models.

### *4.3. Population size of major urban agglomerations*

Barranco-Chamorro and Jiménez-Gamero (2012) and Bedbur et al. (2019) have applied Pareto distribution to the modelling of the size of the hundred (100) largest urban agglomerations in the world at the respective reference dates 01/01/2010 and 01/01/2016. The data are provided by Brinkhoff (2024) through the website `https://www.citypopulation.de/en/world/agglomerations/`. We will compare the different distributions considered in this paper for fitting the population size of the hundred (100) largest urban agglomerations in the world at the reference date 01/01/2024. The descriptive statistics for this dataset are reported in Table 6.

#### 4.3.1. Fitting with the initial (unmodified) distributions

Parameter estimates for the unmodified distributions are reported in Table 7 and the different criteria for the unmodified models are reported in Table 8.

**Table 7.** Parameters estimated for unmodified *CTP* distributions on the Population size dataset

| Distributions | Estimations | | | |
|---|---|---|---|---|
| $CTP_G$ | $x_0 = 5300000$ | $\hat{\alpha} = 1.884$ | $\hat{\lambda}_1 = 0.534$ | $\hat{\lambda}_2 = 0.7$ |
| $CTP_A$ | $x_0 = 5300000$ | $\hat{\alpha} = 1.513$ | $\hat{\lambda} = -0.455$ | |
| $CTP_{R18a}$ | $x_0 = 5300000$ | $\hat{\alpha} = 0.949$ | $\hat{\lambda}_1 = 0.003$ | $\hat{\lambda}_2 = 0.997$ |
| $CTP_{R18b}$ | $x_0 = 5300000$ | $\hat{\alpha} = 1.884$ | $\hat{\lambda}_1 = -0.766$ | $\hat{\lambda}_2 = 0.3$ |
| $CTP_{R19}$ | $x_0 = 5300000$ | $\hat{\alpha} = 1.295$ | $\hat{\lambda} = 0.391$ | |
| $CTP_{R23}$ | $x_0 = 5300000$ | $\hat{\alpha} = 0.969$ | $\hat{\lambda} = 1$ | $\hat{\eta} = 1.102$ |
| *TP* | $x_0 = 5300000$ | $\hat{\alpha} = 1.787$ | $\hat{\lambda} = -0.526$ | |
| Pareto | $x_0 = 5300000$ | $\hat{\alpha} = 1.42$ | | |

**Table 8.** Criteria for comparing unmodified models on the population size dataset

| Distributions | $-\log L^*$ | *AIC* | *AICC* | *BIC* |
|---|---|---|---|---|
| $CTP_{R23}$ | $1681.333^{(1)}$ | $3368.666^{(2)}$ | $3368.916^{(3)}$ | $3376.482^{(5)}$ |
| $CTP_{R18a}$ | $1681.370^{(2)}$ | $3368.740^{(4)}$ | $3368.990^{(4)}$ | $3376.556^{(6)}$ |
| $CTP_G$ | $1681.422^{(3)}$ | $3368.844^{(5)}$ | $3369.094^{(5)}$ | $3376.660^{(7)}$ |
| $CTP_{R18b}$ | $1681.422^{(3)}$ | $3368.844^{(5)}$ | $3369.094^{(5)}$ | $3376.660^{(7)}$ |
| *TP* | $1681.578^{(5)}$ | $3367.156^{(1)}$ | $3367.280^{(1)}$ | $3372.366^{(2)}$ |
| $CTP_A$ | $1682.365^{(6)}$ | $3368.730^{(3)}$ | $3368.854^{(2)}$ | $3373.940^{(3)}$ |
| $CTP_{R19}$ | $1682.764^{(7)}$ | $3369.528^{(8)}$ | $3369.652^{(8)}$ | $3374.738^{(4)}$ |
| Pareto | $1683.638^{(8)}$ | $3369.276^{(7)}$ | $3369.317^{(7)}$ | $3371.881^{(1)}$ |

According to the results in Table 8, it seems that the distribution that best fits the population size dataset is $CTP_{R23}$. In this case, the estimated parameters $\hat{\lambda}_1$ and $\hat{\lambda}_2$ for the $CTP_G$ are positive so the latter is a probability distribution in the rigorous sense of the term.

### 4.3.2. Fitting with modified distributions

Table 9 gives the estimated values of the parameters of the modified models and Table 10 gives the different comparison criteria.

A comparison between Tables 7 and 9 allows us to notice that the parameters of the $CTP_G$, $CTP_{R18a}$ and $CTP_{R18b}$ distributions have changed and the distributions which best fit the data are $CTP_{R23}$, $CTP_{MG}$, $CTP_{MR18a}$ and $CTP_{MR18b}$. The corrections made in this paper have improved the $CTP_G$, $CTP_{R18a}$ and $CTP_{R18b}$ distributions.

Figure 7 shows the histogram of Population size data and fitted probability density functions (*pdfs*) of the modified distributions.

**Table 9.** Estimated parameters for modified *CTP* models on the Population size dataset

| Distributions | Estimations | | | |
|---|---|---|---|---|
| $CTP_{MG}$ | $x_0 = 5300000$ | $\hat{\alpha} = 0.969$ | $\hat{\lambda}_1 = 0.898$ | $\hat{\lambda}_2 = 2.102$ |
| $CTP_{MA}$ | $x_0 = 5300000$ | $\hat{\alpha} = 1.513$ | $\hat{\lambda} = -0.455$ | |
| $CTP_{MR18a}$ | $x_0 = 5300000$ | $\hat{\alpha} = 0.969$ | $\hat{\lambda}_1 = -0.103$ | $\hat{\lambda}_2 = 1.102$ |
| $CTP_{MR18b}$ | $x_0 = 5300000$ | $\hat{\alpha} = 0.969$ | $\hat{\lambda}_1 = 1$ | $\hat{\lambda}_2 = -1.102$ |
| $CTP_{MR19}$ | $x_0 = 5300000$ | $\hat{\alpha} = 1.295$ | $\hat{\lambda} = 0.391$ | |
| $CTP_{R23}$ | $x_0 = 5300000$ | $\hat{\alpha} = 0.969$ | $\hat{\lambda} = 1$ | $\hat{\eta} = 1.102$ |
| *TP* | $x_0 = 5300000$ | $\hat{\alpha} = 1.787$ | $\hat{\lambda} = -0.526$ | |
| Pareto | $x_0 = 5300000$ | $\hat{\alpha} = 1.42$ | | |

**Table 10.** Criteria for modified *CTP* models on the Population size dataset (ranks in parentheses)

| Distributions | $-\log L^*$ | *AIC* | *AICC* | *BIC* |
|---|---|---|---|---|
| $CTP_{MG}$ | $1681.333^{(1)}$ | $3368.666^{(2)}$ | $3368.916^{(3)}$ | $3376.482^{(5)}$ |
| $CTP_{MR18a}$ | $1681.333^{(1)}$ | $3368.666^{(2)}$ | $3368.916^{(3)}$ | $3376.482^{(5)}$ |
| $CTP_{MR18b}$ | $1681.333^{(1)}$ | $3368.666^{(2)}$ | $3368.916^{(3)}$ | $3376.482^{(5)}$ |
| $CTP_{R23}$ | $1681.333^{(1)}$ | $3368.666^{(2)}$ | $3368.916^{(3)}$ | $3376.482^{(5)}$ |
| *TP* | $1681.578^{(5)}$ | $3367.156^{(1)}$ | $3367.280^{(1)}$ | $3372.366^{(2)}$ |
| $CTP_{MA}$ | $1682.365^{(6)}$ | $3368.730^{(6)}$ | $3368.854^{(2)}$ | $3373.940^{(3)}$ |
| $CTP_{MR19}$ | $1682.764^{(7)}$ | $3369.528^{(8)}$ | $3369.652^{(8)}$ | $3374.738^{(4)}$ |
| Pareto | $1683.638^{(8)}$ | $3369.276^{(7)}$ | $3369.317^{(7)}$ | $3371.881^{(1)}$ |

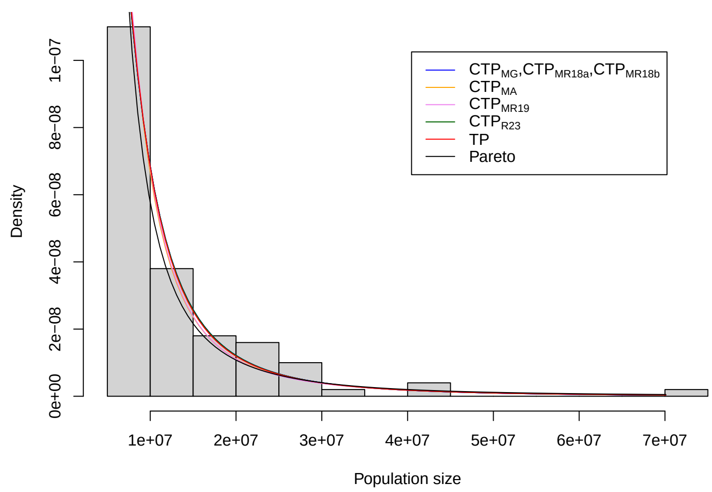


**Fig. 7.** Graphical comparison of the fitting of *CTP* densities to the Population size dataset

## 5. Discussion and concluding remarks

The extension of probability distributions is a broad field of research that has gained significant importance over the last two decades, attracting considerable interest from researchers. In this paper, we focused on the transmutation of probability distributions, one of the most widely used techniques for extending probability models.

The first form of transmutation, the quadratic transmutation introduced by Shaw and Buckley (2009), is unique. However, the cubic transmutation developed later is not, leading to multiple formulations in the literature. In this study, we provided a unified definition of cubic transmutation and examined six cubic transmutation formulas proposed by Granzotto et al. (2017), AL-Kadim and Mohammed (2017), Rahman et al. (2018a), Rahman et al. (2018b), Rahman et al. (2019b) and Rahman et al. (2023). These transmuted cubic distributions involve one or two additional parameters, subject to box and/or linear inequality constraints, in addition to the parameters of the baseline distribution.

Our theoretical and empirical analyses revealed that the cubic transmutation formula of Granzotto et al. (2017) ($CTP_G$) is not always well defined. Specifically, the constraints imposed on the additional parameters do not always ensure that the resulting cumulative distribution function is valid. We demonstrated this issue both theoretically and through empirical illustrations using two real datasets.

To address these limitations, we proposed modified versions of the models by Granzotto et al. (2017), AL-Kadim and Mohammed (2017), Rahman et al. (2018a), Rahman et al. (2018b) and Rahman et al. (2019b), adjusting the parameter ranges to ensure well-defined cumulative distribution functions. Additionally, we showed that by refining these constraints, the cubic transmutation formulas of Granzotto et al. (2017), Rahman et al. (2018a) and Rahman et al. (2018b) become equivalent.

Finally, our real-data applications, using the Pareto distribution as the baseline, confirmed our theoretical findings. The modified models systematically outperformed their unmodified counterparts, reinforcing the necessity of using the improved versions proposed in this paper.

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