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A Framework for Classes of Gamma Mixtures based on Generalized Inverse Gaussian (GIG) distribution

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Abstract. The main objective of this article is to construct a Framework on classes of mixing distributions based on the Generalized Inverse Gaussian distribution. The classes can be used to study any continuous mixture. We propose Gamma , Weighted Inverse Gaussian, and finite mixtures; as classes of mixing distributions. The finite mixtures are also weighted distributions. Mathematical formulations for the mixing distributions are discussed. An application to gamma mixtures has been done. A tractable and indirect method of obtaining the properties of gamma mixtures has been examined. Forms of gamma mixtures have mainly been expressed in terms of special functions. The special functions are gamma function, beta function, modified Bessel function of the third kind, and Tricomi confluent hypergeometric function. Other gamma mixtures have been expressed in explicit form. A general formulation based on the expectation maximization (EM) algorithm for maximum likelihood estimation has also been discussed.

Key words: generalized inverse gaussian; mixing distribution; Bessel function; confluent hyper-geometric function; EM-algorithm.
AMS 2010 Mathematics Subject Classification Objects : 62P05; 60E05; 62E10; 62E15

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Résumé (French Abstract) L'objectif principal de cet article est de construire un cadre pour les classes de distributions de mélange basées sur la distribution généralisée inverse gaussienne. Ces classes peuvent être utilisées pour étudier tout mélange continu. Nous proposons les distributions Gamma, Gaussienne inverse pondérée et les mélanges finis comme classes de distributions de mélange. Les mélanges finis sont également considérés comme des distributions pondérées. Les formulations mathématiques des distributions de mélange sont abordées. Une application aux mélanges gamma a été réalisée. Une méthode indirecte et exploitable pour obtenir les propriétés des mélanges gamma a été examinée. Les formes des mélanges gamma ont principalement été exprimées en termes de fonctions spéciales. Ces fonctions spéciales incluent la fonction gamma, la fonction bêta, la fonction de Bessel modifiée de troisième espèce et la fonction hypergéométrique confluente de Tricomi. D'autres mélanges gamma ont été exprimés sous une forme explicite. Une formulation générale basée sur l'algorithme d'espérance-maximisation (EM) pour l'estimation du maximum de vraisemblance a également été discutée.

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1. Introduction

Wakoli (2016) constructed Exponential mixtures; Sarguta (2017) Poisson mixtures and Otwande (2018) proposed Binomial mixtures and Maina (2022) Normal weighted variance mean mixtures. All these scholars selected mixing distributions in an arbitrary manner. There is need to develop a well structured Framework for classifying mixing distributions.

Generalized Inverse Gaussian (GIG) distribution is denoted as $GIG(\nu, \delta, \gamma)$. Inverse Gaussian distribution denoted by $GIG(-\frac{1}{2}, \delta, \gamma)$ and reciprocal inverse Gaussian distribution denoted by $GIG(\frac{1}{2}, \delta, \gamma)$ are special cases of $GIG(\nu, \delta, \gamma)$ distribution. Other members are $GIG(-\frac{3}{2}, \delta, \gamma)$, $GIG(\frac{3}{2}, \delta, \gamma)$, $GIG(-\frac{5}{2}, \delta, \gamma)$, $GIG(\frac{5}{2}, \delta, \gamma)$ and so on. $GIG(-\frac{1}{2}, \delta, \gamma)$ and $GIG(\frac{1}{2}, \delta, \gamma)$ have been used as mixing distributions by

Sarguta (2017) in Poisson mixtures, Wakoli (2016) in Exponential mixtures and by Maina (2022) in Normal mixtures. $GIG(-\frac{3}{2}, \delta, \gamma)$ and $GIG(\frac{3}{2}, \delta, \gamma)$ have not been used as mixing distributions, except by Maina (2022) who used them in Normal mixtures. Finite mixtures of weighted inverse Gaussian distributions are also weighted inverse Gaussian distributions (see Jorgensen (1991)); Gupta (2011) studied a finite mixture of inverse Gaussian distributions. Maina (2022) used six pairwise finite mixtures of weighted inverse Gaussian distribution as mixing distributions in normal mixtures.

A general framework for studying continuous mixtures is proposed. A continuous mixture is based on a bivariate distribution with its associated conditional and mixing distributions.

Consider a random variable X whose probability density function *pdf* or probability mass function *pmf* is $f(x;\theta)$, where θ is a parameter. Usually θ is fixed. However, there are situations where θ is varying and can therefore be considered as a random variable. So we now have two variables X and Θ taking values x and θ, respectively. Their joint (bi-variate) distribution is denoted by f(x, θ), where θ is assumed to be continuous.

$$\therefore f(x) = \int_0^\infty f(x|\theta)g(\theta)d\theta \tag{1}$$

The marginal distribution $f(x)$ is called a mixture or a mixed distribution. The other marginal distribution $g(\theta)$ is called a mixing distribution. Since θ is continuous, $g(\theta)$ is a continuous mixing distribution and hence $f(x)$ is a continuous mixture.

The conditional distribution can either be discrete or continuous; i.e., $f(x|\theta)$ can either be a *pmf* or **pdf**. If $f(x|\theta)$ is a PMF, then $f(x)$ is a discrete mixture of a continuous distribution. If $f(x|\theta)$ is a *pdf*, then $f(x)$ is a continuous mixture of a continuous distribution.
In the proposed framework, a procedure for identifying mixing distributions for any given continuous conditional distribution is elaborated. A key interest is in the Generalized Inverse Gaussian (*GIG*) distribution considered as a mixing distribution. *GIG* nests a number of standard distributions which are also divided into two classes. The two classes are the limiting and special cases. The limiting cases form the Gamma class while the special cases form the Weighted Inverse Gaussian class.

The remainder of this article is organized as in the following. In Section 2, the Generalized Inverse Gaussian distribution is derived. Section 3 deals with the limiting class of *GIG* as mixing distributions. Weighted Inverse Gaussian mixing class which are the special cases of *GIG* and the main results are analyzed in Section 4. In Section 5, finite mixtures as mixing distributions are elaborated. An application of the Framework to Gamma mixtures and the Expectation Maximization (EM) algorithm are discussed in section 6. The conclusion of this work is given in section 7 where the results are summarized and extensions are suggested.

2. Generalized Inverse Gaussian (*GIG*) distribution

Generalized inverse Gaussian (*GIG*) distribution is based on the Modified Bessel function of the third kind which is denoted by $K_\nu(\omega)$ and defined as

$$K_\nu(\omega) = \frac{1}{2}\int_0^\infty x^{\nu-1}e^{-\frac{\omega}{2}(x+\frac{1}{x})}dx$$

Various parameterizations and transformations have been used to construct *GIG* distribution. We have *GIG* based on Tweedie (1957) parameterization and transformation. Gómez-Deniz (2013) used Tweedie's parameterization to construct Gamma-*GIG* distribution in general and to study Gamma-Inverse Gaussian mixture in particular. Bhattacharya (1986) studied type II Exponential-Inverse Gaussian mixture. The Inverse Gaussian mixing distribution is based in Tweedie's parameterization. Sichel (1971) came up with a parameterization and transformation used in constructing Poisson-*GIG* mixture popularly known as Sichel distribution.

Barndorff-Nielsen (1974) modified Sichel (1971)'s approach and used it to construct a Normal Variance mean mixture with this *GIG* mixing distribution resulting to what is called Generalized Hyperbolic distribution. In this study we shall use Barndorff-Nielsen (1974) approach as described below.
Let

$$\omega = \delta\gamma \ \text{ and } \ x = \frac{\gamma}{\delta}\lambda \ .$$

Therefore the *pdf* of *GIG* is given by

$$g(\lambda) = \frac{\left(\frac{\gamma}{\delta}\right)^\nu \lambda^{\nu-1}e^{-\frac{1}{2}(\gamma^2\lambda+\frac{\delta^2}{\lambda})}}{2K_\nu(\delta\gamma)} \tag{2}$$

$$= \frac{\lambda^{\nu-1}e^{-\frac{1}{2}(\gamma^2\lambda+\frac{\delta^2}{\lambda})}}{\int_0^\infty \lambda^{\nu-1}e^{-\frac{1}{2}(\gamma^2\lambda+\frac{\delta^2}{\lambda})}d\lambda} \tag{3}$$

for $\lambda > 0$; $-\infty < \nu < \infty, \delta > 0, \gamma > 0$. This is a 3-parameter distribution denoted by

$$\Lambda \sim \text{GIG}(\nu, \delta, \gamma) \ .$$

The r^{th} moment is

$$E(\Lambda^r) = \left(\frac{\delta}{\gamma}\right)^r \frac{K_{\nu+r}(\delta\gamma)}{K_\nu(\delta\gamma)}$$

for $-\infty < r < \infty$.

GIG distribution forms a class of its own. It also nests a number of distributions namely; limiting cases and special cases. Gamma and inverse Gamma distributions are the limiting cases belonging to the Gamma mixing class . Inverse Gaussian and Reciprocal Inverse Gaussian distributions are some of the special cases forming weighted Inverse Gaussian mixing class. Hyperbolic distribution is another special case of *GIG* distribution.

3. Gamma mixing class

When $\delta = 0$ in (3) we get gamma distribution denoted by GIG$(\nu, 0, \gamma)$. The *pdf* is given by

$$g(\lambda) = \frac{\left(\frac{\gamma^2}{2}\right)^\nu \lambda^{\nu-1} e^{-\frac{\gamma^2}{2}\lambda}}{\Gamma(\nu)}$$

for $\lambda > 0$; $\nu > 0, \gamma > 0$. Gamma distribution and its generalizations belong to the gamma mixing class.
When $\gamma = 0$ in (3) we get Inverse Gamma distribution denoted by GIG$(\nu, \delta, 0)$. The *pdf* is given by

$$g(\lambda) = \frac{\left(\frac{\delta^2}{2}\right)^\beta}{\Gamma(\beta)} \lambda^{-\beta-1} e^{-\frac{\delta^2}{2\lambda}}$$

for $\lambda > 0$; $\beta = -\nu, \delta > 0, \nu < 0$. Inverse Gamma and its generalizations also belong to the Gamma mixing class. Other members of the Gamma mixing class are Gamma type II, Shifted Gamma and truncated gamma distributions.

4. Weighted Inverse Gaussian (WIG) mixing class

4.1. Inverse Gaussian (IG)distribution

When $\nu = -\frac{1}{2}$ in (2), we get GIG$(-\frac{1}{2}, \delta, \gamma)$ which gives us Inverse Gaussian distribution. The GIG$(-\frac{1}{2}, \delta, \gamma)$ *pdf* is given by

$$g_1(\lambda) = \frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}} \lambda^{-\frac{3}{2}} e^{-\frac{1}{2}(\gamma^2\lambda + \frac{\delta^2}{\lambda})}$$

for $\lambda > 0$; $\delta > 0, \gamma > 0$.

$$E(\Lambda^r) = \left(\frac{\delta}{\gamma}\right)^r \frac{K_{-\frac{1}{2}+r}(\delta\gamma)}{K_{-\frac{1}{2}}(\delta\gamma)}$$

for $-\infty < r < \infty$.

4.2. Weighted distributions

Let Λ be a random variable with *pdf* $g(\lambda)$. Then a function of Λ, say $w(\lambda)$ is also a random variable with expectation

$$E[w(\Lambda)] = \int_0^\infty w(\lambda)g(\lambda)d\lambda$$
$$\therefore 1 = \int_0^\infty \frac{w(\lambda)}{E[w(\Lambda)]} g(\lambda)d\lambda \ .$$

Thus we have a weighted distribution whose *pdf* is given by

$$g_W(\lambda) = \frac{w(\lambda)}{E[w(\Lambda)]} g(\lambda), \ -\infty < \lambda < \infty \ .$$

The concept of a weighted distribution was introduced by Fisher (1934). It was initially elaborated by Rao (1965) and further elaborated by Patil (1978).

4.3. Reciprocal Inverse Gaussian (RIG) distribution

When $\nu = \frac{1}{2}$ in (2) we get GIG$(\frac{1}{2}, \delta, \gamma)$ which gives us the Reciprocal Inverse Gaussian (RIG) distribution written as

$$g_2(\lambda) = \frac{\gamma e^{\delta\gamma}}{\sqrt{2\pi}} \lambda^{-\frac{1}{2}} e^{-\frac{1}{2}(\gamma^2\lambda + \frac{\delta^2}{\lambda})}$$

for $\lambda > 0$; $\delta > 0, \gamma > 0$. So

$$\frac{g_2(\lambda)}{g_1(\lambda)} = \frac{\gamma}{\delta}\lambda \ .$$

$$\therefore g_2(\lambda) = \frac{\gamma}{\delta}\lambda \, g_1(\lambda)$$
$$= \frac{w(\lambda)}{E[w(\Lambda)]} g_1(\lambda) \ .$$

Thus, GIG$(\frac{1}{2}, \delta, \gamma)$ is a weighted Inverse Gaussian distribution with weight function

$$w(\lambda) = \lambda$$
and
$$E[w(\Lambda)] = E(\Lambda) = \frac{\delta}{\gamma} \ .$$

4.4. GIG$(-\frac{3}{2}, \delta, \gamma)$

The *pdf* of GIG$(-\frac{3}{2}, \delta, \gamma)$ is

$$g_3(\lambda) = \frac{\delta^2}{1+\delta\gamma}\frac{1}{\lambda}\, g_1(\lambda) \ .$$

Thus , GIG$(-\frac{3}{2}, \delta, \gamma)$ is a weighted Inverse Gaussian distribution with weight function

$$w(\lambda) = \frac{1}{\lambda}$$
and
$$E[w(\Lambda)] = \frac{1+\delta\gamma}{\delta^2}$$

where

$$\Lambda \sim \text{GIG}(-\frac{3}{2}, \delta, \gamma) \ .$$

4.5. GIG$(\frac{3}{2}, \delta, \gamma)$

The *pdf* of GIG$(\frac{3}{2}, \delta, \gamma)$ is

$$g_4(\lambda) = \frac{\gamma^3}{\delta(1+\delta\gamma)}\, \lambda^2\, g_1(\lambda) \ .$$

Thus, GIG$(\frac{3}{2}, \delta, \gamma)$ is a weighted Inverse Gaussian distribution with weight function

$$w(\lambda) = \lambda^2$$
and
$$E[w(\Lambda)] = \frac{\delta(1+\delta\gamma)}{\gamma^3} \ .$$

Remark 1. GIG$(-\frac{1}{2},\delta,\gamma)$,GIG$(\frac{1}{2},\delta,\gamma)$, GIG$(-\frac{3}{2},\delta,\gamma)$ and GIG$(\frac{3}{2},\delta,\gamma)$ belong to the weighted Inverse Gaussian mixing class.

The Inverse Gaussian mixing distribution GIG$(-\frac{1}{2},\delta,\gamma)$ has been widely used followed by the Reciprocal Inverse Gaussian mixing distribution. In only few cases do we have GIG$(-\frac{3}{2},\delta,\gamma)$ and GIG$(\frac{3}{2},\delta,\gamma)$ being used as mixing distributions. MaMaina (2022) used GIG$(-\frac{3}{2},\delta,\gamma)$ and GIG$(\frac{3}{2},\delta,\gamma)$ as mixing distributions in Normal variance mean mixture.

5. Finite mixtures as mixing distributions

Let $g_i(\lambda)$ for $i = 1,2,3,\cdots,k$ be k distributions or k components of a distribution. Then a finite mixture of g_i's is defined as

$$g(\lambda) = \sum_{i=1}^{k} \omega_i g_i(\lambda)$$

where

$$\omega_i > 0 \text{ and } \sum_{i=1}^{k} \omega_i = 1 \ .$$

For a two-component finite mixture, $k = 2$.

$$\begin{aligned}\therefore g(\lambda) &= \sum_{i=1}^{2} \omega_i g_i(\lambda) \\ &= \omega_1 g_1(\lambda) + \omega_2 g_2(\lambda) \\ &= \omega g_1(\lambda) + (1-\omega) g_2(\lambda)\end{aligned}$$

where $\omega_1 = \omega \Rightarrow \omega_2 = 1 - \omega$.

5.1. Finite mixtures of WIG variables

We now wish to determine two-component finite mixtures from the WIG variables described above. The finite mixtures are also found to be weighted Inverse Gaussian distributions which can be used as mixing distributions.

5.1.1. A finite mixture of GIG$(-\frac{1}{2}, \delta, \gamma)$ and GIG$(\frac{1}{2}, \delta, \gamma)$

Let $g_1(\lambda)$ and $g_2(\lambda)$ be the *pdf*s of Inverse Gaussian distribution and Reciprocal Inverse Gaussian distribution, respectively. The *pdf* of the finite mixture is

$$\begin{aligned} g_{12}(\lambda) &= \omega g_1(\lambda) + (1-\omega) g_2(\lambda) \\ &= \left[\omega + (1-\omega)\frac{\gamma}{\delta}\lambda\right] g_1(\lambda) \end{aligned}$$

Let

$$\omega = \frac{\gamma}{\gamma+\delta}$$

Then

$$g_{12}(\lambda) = \frac{\gamma}{\gamma+\delta}(1+\lambda) g_1(\lambda) \, .$$

Hence the finite mixture of IG and RIG is a weighted Inverse Gaussian distribution with weight function

$$\begin{aligned} w(\lambda) &= 1 + \lambda \\ \text{and} \quad E[w(\Lambda)] &= 1 + E(\Lambda) \\ &= 1 + \frac{\delta}{\gamma} \\ &= \frac{\gamma+\delta}{\gamma} \, . \end{aligned}$$

5.1.2. A finite mixture of GIG$(-\frac{1}{2}, \delta, \gamma)$ and GIG$(-\frac{3}{2}, \delta, \gamma)$

The *pdf* of the finite mixture is

$$\begin{aligned} g_{13}(\lambda) &= \omega g_1(\lambda) + (1-\omega) g_3(\lambda) \\ &= \left[\omega + (1-\omega)\frac{\delta^2}{1+\delta\gamma} * \frac{1}{\lambda}\right] g_1(\lambda) \end{aligned}$$

Let

$$\omega = \frac{\delta^2}{1+\delta^2}$$

$$\therefore g_{13}(\lambda) = \frac{\delta^2}{1+\delta^2}[1 + \frac{1}{1+\delta\gamma} * \frac{1}{\lambda}] g_1(\lambda)$$

which is a weighted Inverse Gaussian distribution with weight function

$$
\begin{aligned}
w(\lambda) &= 1 + \frac{1}{1+\delta\gamma} * \frac{1}{\lambda} \\
\text{and} \quad E[w(\Lambda)] &= 1 + \frac{1}{1+\delta\gamma} E(\frac{1}{\Lambda}) \\
&= \frac{1+\delta^2}{\delta^2}
\end{aligned}
$$

5.1.3. A finite mixture of GIG($-\frac{1}{2}, \delta, \gamma$) and GIG($\frac{3}{2}, \delta, \gamma$)

$$
\begin{aligned}
g_{14}(\lambda) &= \omega g_1(\lambda) + (1-\omega) g_4(\lambda) \\
&= \omega g_1(\lambda) + (1-\omega) \frac{\gamma^3}{\delta(1+\delta\gamma)} \lambda^2 \\
&= \left[\omega + (1-\omega) \frac{\gamma^3}{\delta(1+\delta\gamma)} \lambda^2\right] g_1(\lambda)
\end{aligned}
$$

Let

$$
\omega = \frac{\gamma^3}{\delta + \gamma^3}
$$

Then

$$
\begin{aligned}
g_{14}(\lambda) &= \left[\frac{\gamma^3}{\delta+\gamma^3} + \frac{\delta}{\delta+\gamma^3} * \frac{\gamma^3}{\delta(1+\delta\gamma)} \lambda^2\right] g_1(\lambda) \\
&= \frac{\gamma^3}{\delta+\gamma^3} \left(1 + \frac{\lambda^2}{1+\delta\gamma}\right) g_1(\lambda)
\end{aligned}
$$

which is a weighted Inverse Gaussian distribution with weight function

$$
\begin{aligned}
w(\lambda) &= 1 + \frac{\lambda^2}{1+\delta\gamma} \\
\text{and} \quad E[w(\Lambda)] &= 1 + \frac{E(\Lambda^2)}{1+\delta\gamma} \\
&= \frac{\delta+\gamma^3}{\gamma^3}
\end{aligned}
$$

5.1.4. A finite mixture of GIG($\frac{1}{2}, \delta, \gamma$) and GIG($-\frac{3}{2}, \delta, \gamma$)

$$\begin{aligned} g_{23}(\lambda) &= \omega g_2(\lambda) + (1-\omega) g_3(\lambda) \\ &= \omega \frac{\gamma}{\delta} \lambda g_1(\lambda) + (1-\omega) \frac{\delta^2}{1+\delta\gamma} * \frac{1}{\lambda} g_1(\lambda) \\ g_{23}(\lambda) &= \left[\frac{\omega\gamma}{\delta} \lambda + (1-\omega) \frac{\delta^2}{1+\delta\gamma} * \frac{1}{\lambda} \right] g_1(\lambda) \end{aligned}$$

Put

$$\omega = \frac{\delta^3}{\delta^3 + \gamma}$$

$$\therefore g_{23}(\lambda) = \frac{\delta^2\gamma}{\delta^3+\gamma} [\lambda + \frac{1}{1+\delta\gamma} * \frac{1}{\lambda}] g_1(\lambda)$$

which is a weighted Inverse Gaussian distribution with weight function

$$\begin{aligned} w(\lambda) &= \lambda + \frac{1}{1+\delta\gamma} * \frac{1}{\lambda} \\ \text{and} \quad E[w(\Lambda)] &= \frac{\delta^3+\gamma}{\gamma\delta^2} \end{aligned}$$

5.1.5. A finite mixture of GIG($\frac{1}{2}, \delta, \gamma$) and GIG($\frac{3}{2}, \delta, \gamma$)

$$\begin{aligned} g_{24}(\lambda) &= \omega g_2(\lambda) + (1-\omega) g_4(\lambda) \\ &= \left[\frac{\omega\gamma}{\delta} \lambda + \frac{(1-\omega)\gamma^3}{\delta(1+\delta\gamma)} \lambda^2 \right] g_1(\lambda) \end{aligned}$$

Put

$$\omega = \frac{\gamma^2}{\gamma^2+1}$$

Then

$$g_{24}(\lambda) = \frac{\gamma^3}{\delta(1+\gamma^2)} \left(\lambda + \frac{\lambda^2}{1+\delta\gamma} \right) g_1(\lambda)$$

which is a weighted Inverse Gaussian distribution with weight function

$$\begin{aligned} w(\lambda) &= \lambda + \frac{\lambda^2}{1+\delta\gamma} \\ \text{and} \quad E[w(\Lambda)] &= E(\Lambda) + \frac{E(\Lambda^2)}{1+\delta\gamma} \\ &= \frac{\delta(\gamma^2+1)}{\gamma^3} \end{aligned}$$

5.1.6. A finite mixture of GIG$(-\frac{3}{2},\delta,\gamma)$ and GIG$(\frac{3}{2},\delta,\gamma)$

$$\begin{aligned}g_{34}(\lambda) &= \omega g_3(\lambda)+(1-\omega)g_4(\lambda)\\ &= \left[\frac{\omega\delta^2}{1+\delta\gamma}*\frac{1}{\lambda}+\frac{(1-\omega)\gamma^3}{\delta(1+\delta\gamma)}\lambda^2\right]g_1(\lambda)\end{aligned}$$

Put

$$\omega=\frac{\gamma^3}{\delta^3+\gamma^3}$$

$$\therefore g_{34}(\lambda)=\frac{\delta^3\gamma^3}{(\delta^3+\gamma^3)(1+\delta\gamma)}\left(\frac{1}{\lambda}+\lambda^2\right)g_1(\lambda)$$

which is a weighted Inverse Gaussian distribution with weight function

$$\begin{aligned}w(\lambda) &= \frac{1}{\lambda}+\lambda^2\\ \text{and}\quad E[w(\Lambda)] &= E\left(\frac{1}{\lambda}\right)+E(\Lambda^2)\\ &= \frac{(1+\delta\gamma)(\gamma^3+\delta^3)}{\delta^2\gamma^3}\end{aligned}$$

5.2. Finite mixtures based on Gamma mixing class

5.2.1. One-parameter Lindley distribution

Let $X_1\sim\exp(\theta)=\text{Gamma}(1,\theta)$ and $X_2\sim\text{Gamma}(2,\theta)$. Then the *pdf* of the finite mixture is given by

$$\begin{aligned}f(x) &= \omega f_1(x)+(1-\omega)f_2(x)\\ &= \omega\theta e^{-\theta x}+(1-\omega)\frac{\theta^2}{\Gamma(2)}x^{2-1}e^{-\theta x}\\ &= [\omega\theta+(1-\omega)\theta^2x]e^{-\theta x}\end{aligned}$$

Put

$$\omega=\frac{\theta}{1+\theta}$$

$$\therefore f(x)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, x>0;\ \theta>0$$

This is a one-parameter Lindley distribution introduced by Lindley (1958). It can be re-written as

$$f(x) = \frac{\theta}{1+\theta}(1+x)\theta e^{-\theta x}$$

Thus a one-parameter Lindley distribution is a weighted exponential distribution with weight function

$$\begin{aligned} w(x) &= 1+x \\ \text{and}\quad E[w(X)] &= 1+E(X) \\ &= 1+\int_0^\infty x\theta e^{-\theta x}dx \\ &= 1+\theta\int_0^\infty xe^{-\theta x}dx \\ &= 1+\theta\frac{\Gamma(2)}{\theta^2} \\ &= 1+\frac{1}{\theta} \\ &= \frac{1+\theta}{\theta} \end{aligned}$$

5.2.2. Two-Parameter Lindley distribution

Let $X_1 \sim \text{Gamma}(\alpha,\theta)$ and $X_2 \sim \text{Gamma}(\alpha+1,\theta)$. The *pdf* of the finite mixture becomes

$$\begin{aligned} f(x) &= \omega f_1(x) + (1-\omega)f_2(x) \\ &= \omega\frac{\theta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\theta x} + (1-\omega)\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}x^\alpha e^{-\theta x} \\ &= \left[\omega\frac{\theta^\alpha}{\Gamma(\alpha)} + (1-\omega)\frac{\theta^{\theta+1}}{\Gamma(\alpha+1)}x\right]x^{\alpha-1}e^{-\theta x} \end{aligned}$$

Put

$$\omega = \frac{\theta}{1+\theta}$$

$$\therefore f(x) = \frac{\theta^{\alpha+1}}{(1+\theta)\Gamma(\alpha+1)}(\alpha+x)x^{\alpha-1}e^{-\theta x}$$

for $x > 0$; $\alpha > 0, \theta > 0$. This is a two-parameter Lindley distribution obtained by Zakerzadeh and Dalati (2009). It can be re-written as

$$f(x) = \frac{\theta^{\alpha}}{(1+\theta)\Gamma(\alpha+1)}(\alpha+x)x^{\alpha-1}\theta e^{-\theta x}$$

which is a weighted exponential distribution with weight function

$$w(x) = (\alpha+x)x^{\alpha-1}$$
$$\text{and} \quad E[w(X)] = \frac{(1+\theta)\Gamma(\alpha+1)}{\theta^{\alpha}}$$

5.2.3. A general Two-parameter Lindley distribution

Let $X_j \sim \text{Gamma}(c+j-1,\theta), \ \ j=1,2$. The *pdf* of the finite mixture is written as

$$f(x) = \omega f_1(x) + (1-\omega)f_2(x)$$

i.e.,

$$\begin{aligned}
f(x) &= \omega\text{Gamma}(c,\theta) + (1-\omega)\text{Gamma}(c+1,\theta) \\
&= \omega\frac{\theta^c}{\Gamma(c)}e^{-\theta x}x^{c-1} + (1-\omega)\frac{\theta^{c+1}}{\Gamma(c+1)}e^{-\theta x}x^c \\
&= \frac{\omega c\theta^c}{\Gamma(c+1)}e^{-\theta x}x^{c-1} + (1-\omega)\frac{\theta^{c+1}}{\Gamma(c+1)}e^{-\theta x}x^c \\
&= \frac{\omega c\theta^{c+1}e^{-\theta x}x^{c-1}}{\theta\Gamma(c+1)} + (1-\omega)\frac{\theta^{c+1}}{\Gamma(c+1)}e^{-\theta x}x^c \\
&= \frac{\theta^{c+1}}{\Gamma(c+1)}\left[\frac{\omega c}{\theta} + (1-\omega)x\right]x^{c-1}e^{-\theta x}
\end{aligned}$$

Put

$$\omega = \frac{\theta}{\theta+\alpha}$$

$$\begin{aligned}
\therefore f(x) - \frac{\theta^{c+1}}{\Gamma(c+1)}\left[\frac{c\theta}{\theta(\theta+\alpha)} + \frac{\alpha}{\theta+\alpha}x\right]x^{c-1}e^{-\theta x} \\
= \frac{\theta^{c+1}}{\Gamma(c+1)} * \frac{1}{\theta+\alpha}[c+\alpha x]x^{c-1}e^{-\theta x}
\end{aligned}$$

for $x > 0; \ \ \alpha > 0, \theta > 0, c > 0$. This is a Two-component , 3-parameter Lindley distribution. When $\alpha = 1$, we have

$$f(x) = \frac{\theta^{c+1}}{(\theta+1)\Gamma(c+1)}x^{c-1}(c+x)e^{-\theta x}$$

for $x > 0;\ \ \theta > 0, c > 0$, which is a two-component, two-parameter Lindley distribution. When $\alpha = c$, then

$$
\begin{aligned}
f(x) &= \frac{\theta^{c+1}}{(\theta+c)\Gamma(c+1)} x^{c-1}(c+cx)e^{-\theta x} \\
&= \frac{c\theta^{c+1}x^{c-1}(1+x)e^{-\theta x}}{(\theta+c)\Gamma(c+1)} \\
&= \frac{\theta^{c+1}x^{c-1}(1+x)e^{-\theta x}}{(\theta+c)\Gamma(c)} \, .
\end{aligned}
$$

This is another two-component, two-parameter Lindley distribution. When $c = 1$, we get

$$
f(x) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x}, x > 0;\ \ \alpha > 0, \theta > 0\, .
$$

This is a two-component, two-parameter Lindley distribution. When $c = \alpha = 1$, then

$$
f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, x > 0;\ \theta > 0
$$

which is a one-component Lindley distribution.

5.2.4. 3-parameter Lindley distribution

Let $X_j \sim \text{Gamma}(c+j-1,\theta), j = 1,2,3$. The *pdf* of the finite mixture becomes

$$
\begin{aligned}
f(x) &= \omega_1 \text{Gamma}(c,\theta) + \omega_2 \text{Gamma}(c+1,\theta) + \omega_3 \text{Gamma}(c+2,\theta) \\
&= \omega_1 \frac{\theta^c}{\Gamma(c)} x^{c-1}e^{-\theta x} + \omega_2 \frac{\theta^{c+1}}{\Gamma(c+1)} x^c e^{-\theta x} + \omega_3 \frac{\theta^{c+2}}{\Gamma(c+2)} x^{c+1} e^{-\theta x}
\end{aligned}
$$

Put

$$
\omega_1 = \frac{\theta^2}{1+\theta+\theta^2},\ \omega_2 = \frac{\theta}{1+\theta+\theta^2},\ \ \omega_3 = \frac{1}{1+\theta+\theta^2}
$$

$$
\begin{aligned}
\therefore f(x) &= \frac{\theta^{c+2}}{1+\theta+\theta^2}\left\{\frac{xx^{c-1}}{\Gamma(c)} + \frac{x^c}{\Gamma(c+1)} + \frac{x^2}{\Gamma(c+1)}\right\}e^{-\theta x} \\
&= \frac{\theta^{c+2}}{1+\theta+\theta^2}\left\{\frac{1}{\Gamma(c)+\frac{x}{\Gamma(c+1)}} + \frac{x^2}{\Gamma(c+2)}\right\}x^{c-1}e^{-\theta x} \\
&= \frac{\theta^{c+2}}{1+\theta+\theta^2}\left\{\frac{c(c+1)}{\Gamma(c+1)} + \frac{(c+1)x}{\Gamma(c+2)} + \frac{x^2}{\Gamma(c+2)}\right\}x^{c-1}e^{-\theta x} \\
&= \frac{\theta^{c+2}}{(1+\theta+\theta^2)(\Gamma(c+2))}\left\{c(c+1)^2 + (c+1)x + x^2\right\}x^{c-1}e^{-\theta x}
\end{aligned}
$$

This is a 3-component, two-parameter Lindley distribution. It can also be re-written as

$$f(x) = \frac{\theta^{c+1}}{(1+\theta+\theta^2)\Gamma(c+2)}[c(c+1)+(c+1)x+x^2]x^{c-1}\theta e^{-\theta x}$$

which is a weighted exponential distribution with weight function.

$$\begin{aligned} w(x) &= [c(c+1)^2+(c+1)x+x^2]x^{c-1} \\ \text{and} \quad E[w(X)] &= \frac{\Gamma(c+1)}{\theta^{c+1}}\left\{(c+1)^2\theta^2+(c+1)\theta+c+1\right\} \end{aligned}$$

It can also be re-written as

$$f(x) = \frac{1}{(1+\theta+\theta^2)}\left[\frac{c(c+1)^2+(c+1)x+x^2}{x^2}\right]\frac{\theta^{c+2}}{\Gamma(c+2)}e^{-\theta x}x^{c+1}$$

which is a weighted Gamma distribution with weight function.

$$\begin{aligned} w(x) &= 1+\frac{c+1}{x}+\frac{c(c+1)^2}{x^2} \\ \text{and} \quad E[w(X)] &= 1+\theta+(c+1)\theta^2 \end{aligned}$$

Remark 1. Lindley distribution and its generalizations are finite mixtures of gamma distributions and also weighted exponential distributions.

Remark 2. Fig. 1, page 3831. shows a Framework for Classes of mixing distributions based on GIG distribution for Gamma mixtures.

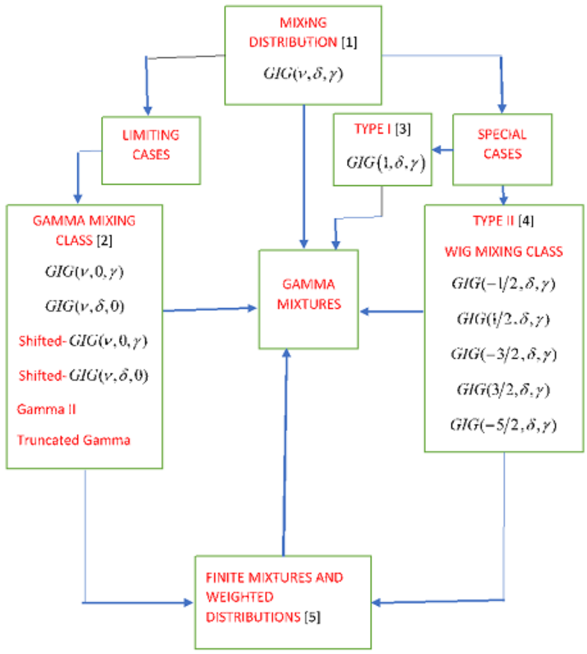

Fig. 1. A Frame Work for classes of Gamma mixtures based on Generalized Inverse Gaussian distribution

Please note that for a specific mixing distribution, various conditional distributions will produce different results. Take a gamma mixing distribution given by

$$g(\lambda) = \frac{\mu^{\beta}}{\Gamma(\beta)}\lambda^{\beta-1}e^{-\mu\lambda}, \lambda > 0; \beta, \mu > 0$$

.

Then for a Poisson mixture

$$f(x) = \binom{\beta + x - 1}{x}\left(\frac{\mu}{1+\mu}\right)^{\beta}\left(\frac{1}{1+\mu}\right)^{x}, \; x = 0, 1, 2, 3, \cdots$$

which is a negative binomial distribution Greenwood (1920). For a Weibull mixture

$$f(x) = \frac{\alpha\beta\mu^{\beta}x^{\alpha-1}}{(\mu + x^{\alpha})^{\beta+1}}, \; x > 0\,; \;\; \alpha, \beta, \mu > 0$$

which is a weibull-Gamma distribution. For a Normal-Gamma I mixture

$$f(x) = \frac{2\mu^\beta}{\sqrt{2\pi}\Gamma(\beta)} \left(\sqrt{\frac{x^2}{2\mu}}\right)^{\beta-\frac{1}{2}} K_{\beta-\frac{1}{2}}(2x\sqrt{\frac{\mu}{2}})$$

which is a variance-Gamma distribution.

6. Gamma mixtures

6.1. General formulation

The general formulation of a Gamma mixture is given by

$$f(x) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} g(\lambda) d\lambda, \; x > 0; \; \alpha, \lambda > 0$$

When $\alpha = 1$ we obtain an exponential mixture given by

$$f(x) = \int_0^\infty \lambda e^{-\lambda x} g(\lambda) d\lambda$$

as given by Wakoli (2016). When $\alpha = n$ we get an Erlang mixture given by

$$f(x) = \int_0^\infty \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} g(\lambda) d\lambda$$

as studied by Gathongo (2019). The integration problem in the formulation of a gamma mixture requires the use of given special functions. In most cases it may be difficult to evaluate the mixture properties directly. A relationship between properties of gamma mixtures and properties of mixing distributions provides a tractable indirect method. The following formulation results are key in obtaining the properties of gamma mixtures indirectly.

$$
\begin{aligned}
M_X(t) &= E[\Lambda^{\alpha}(\Lambda - t)^{-\alpha}] \\
E(X^r) &= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} E\left[\frac{1}{\Lambda^r}\right] \\
E(X) &= \alpha E\left(\frac{1}{\Lambda}\right) \\
E(X^2) &= \alpha(\alpha+1) E\left(\frac{1}{\Lambda^2}\right) \\
E(X^3) &= \alpha(\alpha+1)(\alpha+2) E\left(\frac{1}{\Lambda^3}\right) \\
\text{Var}(X) &= \alpha^2 Var(\frac{1}{\Lambda}) + \alpha E\left(\frac{1}{\Lambda^2}\right) \\
\mu_3(X) &= \alpha(\alpha+1)(\alpha+2)\mu_3\left(\frac{1}{\Lambda}\right) + 6\alpha(\alpha+1)\text{Var}\left(\frac{1}{\Lambda}\right) E\left(\frac{1}{\Lambda}\right) \\
&\quad + 2\alpha[E\left(\frac{1}{\Lambda}\right)]^3
\end{aligned}
$$

6.2. Some forms of gamma mixtures by cases

In this sequel, the conditional distribution is a gamma distribution given by

$$f(x|\lambda) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, x > 0; \quad \alpha, \lambda > 0$$

6.2.1. Gamma I-Gamma I mixture

When the mixing distribution is defined as

$$g(\lambda) = \frac{\mu^{\beta}}{\Gamma(\beta)} \lambda^{\beta-1} e^{-\mu\lambda}, \lambda > 0; \quad \beta, \mu > 0$$

a generalized Pareto is obtained as:

$$f(x) = \frac{\mu^{\beta} x^{\alpha-1}}{B(\alpha,\beta)(x+\mu)^{\alpha+\beta}} \; x > 0; \alpha, \beta, \mu > 0$$

The mixture expression is in terms of a beta function.
The r^{th} moment is

$$E(X^r) = \mu^r \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{\Gamma(\beta-r)}{\Gamma(\beta)}$$

which is in terms of gamma functions. Hence

$$
\begin{aligned}
E(X) &= \frac{\alpha}{\beta - 1}\mu,\ \beta > 0 \\
E(X^2) &= \frac{\alpha(\alpha+1)}{(\beta-1)(\beta-2)}\mu^2,\ \ \beta > 2 \\
Var(X) &= \frac{\alpha(\alpha+\beta-1)}{(\beta-1)^2(\beta-2)}\mu^2 \\
E(X^3) &= \frac{\alpha(\alpha+1)(\alpha+2)}{(\beta-1)(\beta-2)(\beta-3)}\mu^3 \\
E[X-E(X)]^3 &= \frac{2\alpha[2\alpha^2+(\beta-1)(3\alpha+\beta-1)]}{(\beta-1)^3(\beta-2)(\beta-3)}\mu^3 \\
\text{Skewness} &= \frac{2(\beta-2)^{0.5}[2\alpha^2+(\beta-1)(3\alpha+\beta-1)]}{\alpha^{0.5}(\beta-3)(\alpha+\beta-1)^{1.5}}
\end{aligned}
$$

If $\alpha = 1$, then a Pareto distribution is obtained as

$$
f(x) = \frac{\beta\mu^\beta}{(x+\mu)^{\beta+1}},\ \ x > 0; \mu, \beta > 0
$$

6.2.2. Gamma I-Inverse Gamma I mixture

Given

$$
g(\lambda) = \frac{\mu^\beta}{\Gamma(\beta)}\lambda^{-\beta-1}e^{-\frac{\mu}{\lambda}},\ \lambda > 0
$$

$$
f(x) = \frac{2\mu^{\frac{\alpha+\beta}{2}}x^{\frac{\alpha+\beta}{2}-1}}{\Gamma(\alpha)\Gamma(\beta)}K_{\alpha-\beta}(2\sqrt{\mu x}),\ x > 0
$$

where $K_\nu(\cdot)$ is a Modified Bessel function of the third kind.

$$
E(X^r) = \mu^{-r}\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\frac{\Gamma(\beta+r)}{\Gamma(\beta)}
$$

Therefore

$$\begin{aligned}
E(X) &= \frac{\alpha\beta}{\mu} \\
E(X^2) &= \frac{\alpha\beta(\alpha+1)(\beta+1)}{\mu^2} \\
E(X^3) &= \frac{\alpha\beta(\alpha+1)(\beta+1)(\alpha+2)(\beta+2)}{\mu^3} \\
\text{var}(X) &= \frac{\alpha\beta}{\mu^2}(\alpha+\beta+1) \\
E[X-E(X)]^3 &= \frac{\alpha\beta}{\mu^3}\left[\begin{matrix}(\alpha+1)(\beta+1)(\alpha+2)(\beta+2)-3\alpha\beta(\alpha+1)(\beta+1) \\ +2\alpha^2\beta^2\end{matrix}\right] \\
&= \frac{\alpha\beta}{\mu^3}\{(\alpha+1)(\beta+1)[(\alpha+2)(\beta+2)-3\alpha\beta]+2\alpha^2\beta^2\} \\
&= \frac{2\alpha\beta}{\mu^3}\left\{(\alpha\beta+\alpha+\beta+1)(\alpha+\beta+2-\alpha\beta)+\alpha^2\beta^2\right\} \\
&= \frac{2\alpha\beta}{\mu^3}(3\alpha\beta+3\alpha+\alpha^2+\beta^2+3\beta+2) \\
&= \frac{2\alpha\beta}{\mu^3}\left[3\alpha\beta+\alpha+\beta+(\alpha+1)^2+(\beta+1)^2\right] \\
&= \frac{2\alpha\beta}{\mu^3}\left[\alpha\beta+(\alpha+\beta+1)(\alpha+\beta+2)\right] \\
&= \frac{2\alpha\beta}{\mu^3}\left[\alpha\beta+(\alpha+\beta+1)^2+\alpha+\beta+1\right] \\
&= \frac{2\alpha\beta}{\mu^3}\left[(\alpha+\beta+1)^2+\alpha(\beta+1)+\beta+1\right] \\
&= \frac{2\alpha\beta}{\mu^3}\left[(\alpha+\beta+1)^2+(\beta+1)(\alpha+1)\right] \\
\therefore \gamma_1 &= \frac{2\left[(\alpha+\beta+1)^2+(\beta+1)(\alpha+1)\right]}{[\alpha\beta(\alpha+\beta+1)^3]^{\frac{1}{2}}}
\end{aligned}$$

6.2.3. Gamma I -Shifted gamma I mixture

$$g(\lambda) = \frac{\mu^\beta}{\Gamma(\beta)}(\lambda-\theta)^{\beta-1}e^{-(\lambda-\theta)\mu}, \lambda > \theta;\ \ \beta,\mu,\theta > 0$$

$$f(x) = \frac{x^{\alpha-1}\theta^{\beta+\alpha}\mu^\beta}{\Gamma(\alpha)}\psi(\beta,\alpha+\beta+1,\theta x+\theta\mu),\ x > 0;\ \ \alpha,\beta,\mu > 0 \tag{4}$$

where $\psi(\cdot)$ is a Tricomi confluent hypergeometric function.

$$E(X^r) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\theta^{\beta-r}\mu^\beta\psi(\beta,\beta-r+1,\theta\mu)$$

Therefore

$$
\begin{aligned}
E(X) &= \alpha\mu^{\beta}\theta^{\beta-1}\psi(\beta;\beta;\theta\mu) && (5)\\
E(X^2) &= \alpha(\alpha+1)\theta^{\beta-2}\mu^{\beta}\psi(\beta;\beta-1;\theta\mu) && (6)\\
E(X^3) &= \alpha(\alpha+1)(\alpha+2)\theta^{\beta-3}\mu^{\beta}\psi(\beta;\beta-2;\theta\mu) && (7)\\
Var(X) &= \begin{bmatrix}\alpha(\alpha+1)\theta^{\beta-2}\mu^{\beta}\psi(\beta;\beta-1;\theta\mu)\\ -\alpha^2\theta^{2\beta-2}\mu^{2\beta}\psi^2(\beta;\beta;\theta\mu)\end{bmatrix} && (8)\\
E[X-E(X)]^3 &= \alpha\theta^{\beta-3}\mu^{\beta}\begin{bmatrix}(\alpha+1)(\alpha+2)\psi(\beta;\beta-2;\theta\mu)\\ -3\alpha(\alpha+1)\theta^{\beta}\mu^{\beta}\psi(\beta;\beta-1;\theta\mu)\\ \times\psi(\beta;\beta;\theta\mu)\\ +2\alpha^2\theta^{2\beta}\mu^{2\beta}\psi^3(\beta;\beta;\theta\mu)\end{bmatrix} && (9)
\end{aligned}
$$

and

$$
\text{Skewness} = \frac{\begin{bmatrix}(\alpha+1)(\alpha+2)\psi(\beta;\beta-2;\theta\mu)\\ -3\alpha(\alpha+1)\theta^{\beta}\mu^{\beta}\psi(\beta;\beta-1;\theta\mu)\\ \times\psi(\beta;\beta;\theta\mu)\\ +2\alpha^2\theta^{2\beta}\mu^{2\beta}\psi^3(\beta;\beta;\theta\mu)\end{bmatrix}}{\sqrt{\alpha\theta^{\beta}\mu^{\beta}}\left(\sqrt{\begin{bmatrix}(\alpha+1)\psi(\beta;\beta+1;\theta\mu)\\ -\alpha\mu^{\beta}\theta^{\beta}\psi^2(\beta;\beta;\theta\mu)\end{bmatrix}}\right)^3} \tag{10}
$$

6.2.4. Gamma I-Lindley mixture

$$
\begin{aligned}
g(\lambda) &= \frac{\theta^2}{1+\theta}(\lambda+1)e^{-\theta\lambda},\ \lambda>0;\ \theta>0\\
f(x) &= \frac{\alpha\theta^2x^{\alpha-1}(x+\alpha+\theta+1)}{(\theta+1)(x+\theta)^{\alpha+2}}\\
E(X^r) &= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\frac{\theta^r\Gamma(1-r)}{1+\theta}(\theta-r+1)
\end{aligned}
$$

Though the mixture is in explicit form, the rth moment is in terms of gamma functions. For $r=1,2,3,\cdots$; Gamma I-Lindley distribution has no moments.

6.2.5. Gamma I-Generalized Lindley mixture

$$
g(\lambda) = \frac{\theta^2(\theta\lambda)^{\beta-1}(\beta+\lambda)e^{-\theta\lambda}}{(1+\theta)\Gamma(\beta+1)},\quad \lambda>0;\ \theta>0,\beta>0
$$

$$
\begin{aligned}
f(x) &= \frac{\theta^{\beta+1}x^{\alpha+1}}{(1+\theta)\Gamma(\alpha)\Gamma(\beta+1)}\left\{\frac{\beta\Gamma(\alpha+\beta)}{(x+\theta)^{\alpha+\beta}}+\frac{\Gamma(\alpha+\beta+1)}{(x+\theta)^{\alpha+\beta+1}}\right\}\\
&= \frac{\theta^{\beta+1}x^{\alpha-1}}{B(\alpha,\beta)\beta(x+\theta)^{\alpha+\beta+1}}(\beta x+\beta\theta+\alpha+\beta)
\end{aligned}
$$

$$E(X^r) = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{\theta^{\beta+1}\Gamma(\beta-r)}{(1+\theta)\Gamma(\beta+1)\theta^{\beta-r+1}}(\beta\theta+\beta-r)$$

The mixture is either expressed in terms of gamma functions or a beta function while the rth moment has been expressed in terms of gamma functions. Hence

$$\begin{aligned}
E(X) &= \frac{\alpha\theta(\beta\theta+\beta-1)}{\beta(\beta-1)(1+\theta)} \\
\text{Var}(X) &= \frac{\alpha\theta^2}{\beta(\beta-1)}\left[\frac{\beta(\beta-1)(\alpha+1)(\beta\theta+\beta-2)(1+\theta)-(\beta\theta+\beta-1)^2(\beta-2)}{\beta(\beta-1)(\beta-2)(1+\theta^2)}\right] \\
\text{Skewness} &= \frac{\alpha\theta^3}{\beta(\beta-1)(1+\theta)}\left\{\frac{\begin{bmatrix}\beta^2(\beta-1)^2(\alpha+1)(\alpha+2)(\beta\theta+\beta-3)(1+\theta)^2\\ -3\alpha\beta(\alpha+1)(\beta-1)(1+\theta)\\ \times(\beta\theta+\beta-1)(\beta\theta+\beta-2)\\ +2\alpha^2(\beta-2)(\beta-3)(\beta\theta-1)^2\end{bmatrix}}{\beta^2(\beta-1)^2(1+\theta)^2(\beta-2)(\beta-3)}\right\}
\end{aligned}$$

6.2.6. Gamma I-Inverse Gaussian mixture

$$g(\lambda) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma}\lambda^{-\frac{3}{2}} e^{-\frac{1}{2}\left(\gamma^2\lambda+\frac{\delta^2}{\lambda}\right)} \quad -\infty < \lambda < \infty; \;\; \delta,\gamma > 0$$

$$f(x) = \sqrt{\frac{2}{\pi}} e^{\delta\gamma}\frac{x^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\delta}{\sqrt{2x+\gamma^2}}\right)^{\alpha-\frac{1}{2}} K_{\alpha-\frac{1}{2}}(\delta\sqrt{2x+\gamma^2}) \tag{11}$$

$$E(X^r) = \sqrt{\frac{2}{\pi}}\frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}\delta e^{\delta\gamma}\left(\frac{\gamma}{\delta}\right)^{r+\frac{1}{2}} K_{r+\frac{1}{2}}(\delta\gamma)$$

Both the mixture and its rth moment have been expressed in terms of Modified Bessel function of the third kind. Therefore

$$\begin{aligned}
E(X) &= \frac{\alpha(1+\delta\gamma)}{\delta^2} \\
E(X^2) &= \frac{\alpha(\alpha+1)(3+3\delta\gamma+\delta^2\gamma^2)}{\delta^4} \\
E(X^3) &= \frac{\alpha(\alpha+1)(\alpha+2)(15+15\delta\gamma+6\delta^2\gamma^2+\delta^3\gamma^3)}{\delta^6} \\
\text{Var}(X) &= \frac{\alpha(2\alpha+3+2\alpha\delta\gamma+3\delta\gamma+\alpha\delta^2\gamma^2+\delta^2\gamma^2)}{\delta^4} \\
E[X-E(X)]^3 &= \frac{\alpha}{\delta^6}\begin{bmatrix} 8\alpha^2+36\alpha+30+3\alpha^2\delta\gamma+6\alpha^2\delta^2\gamma^2+27\alpha\delta\gamma \\ +6\alpha\delta^2\gamma^2+30\delta\gamma+6\alpha^2\delta^3\gamma^3+12\delta^2\gamma^2+2\delta^3\gamma^3 \end{bmatrix} \\
\text{Skewness} &= \frac{\begin{bmatrix} 8\alpha^2+36\alpha+30+3\alpha^2\delta\gamma+6\alpha^2\delta^2\gamma^2+27\alpha\delta\gamma \\ +6\alpha\delta^2\gamma^2+30\delta\gamma+6\alpha^2\delta^3\gamma^3+12\delta^2\gamma^2+2\delta^3\gamma^3 \end{bmatrix}}{\alpha^{\frac{1}{2}}(2\alpha+3+2\alpha\delta\gamma+3\delta\gamma+\alpha\delta^2\gamma^2+\delta^2\gamma^2)^{\frac{3}{2}}}
\end{aligned}$$

6.2.7. Gamma I-Beta I mixture

$$g(\lambda) = \frac{\lambda^{p-1}(1-\lambda)^{q-1}}{B(p,q)},\ 0<\lambda<1; p,q>0$$

$$f(x) = \frac{B(q,\alpha+p)}{B(p,q)}\frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)}\ {}_1F_1(q;\alpha+p+q;x)$$

$$E(X^r) = \frac{\Gamma(\alpha+r)\Gamma(p-r)\Gamma(p+q)}{\Gamma(\alpha)\Gamma(p+q-r)\Gamma(p)}$$

The mixture is in terms of beta and gamma functions while the rth moment is in terms of gamma functions only. Hence

$$\begin{aligned}
E(X) &= \frac{\alpha(p+q-1)}{(p-1)} \\
\text{Var}(X) &= \frac{\alpha(p+q-1)}{p(p-1)^2(p-2)}\begin{bmatrix} 4\alpha p^2-5\alpha p-3\alpha q+2\alpha+p^2+pq \\ -3p-q-\alpha p^3+2\alpha pq+2 \end{bmatrix} \\
\text{Skewness} &= \frac{\alpha(p+q-1)}{p-1}\begin{bmatrix} \frac{(\alpha+1)(\alpha+2)(p+q-2)(p+q-3)}{p(p-2)(p-3)} \\ -\frac{3\alpha(\alpha+1)(p+q-1)(p+q-2)}{p()p-1}(p-2)+\frac{2\alpha^2(p+q-1)^2}{(p-1)^2} \end{bmatrix}
\end{aligned}$$

6.2.8. Gamma I-Scaled Beta I mixture

$$g(\lambda) = \frac{\lambda^{p-1}(\mu-\lambda)^{q-1}}{\mu^{p+q-1}B(p,q)},\ 0<\lambda<\mu; p,q>0$$

$$f(x) = \frac{\mu^\alpha x^{\alpha+1}e^{-\mu x}B(q,\alpha+p)}{\Gamma(\alpha)B(p,q)}\ {}_1F_1(q;\alpha+p;\mu x)$$

$$E(X^r) = \frac{\Gamma(\alpha+r)\Gamma(p-r)\Gamma(p+q)}{\Gamma(\alpha)\Gamma(p+q-r)\Gamma(p)}\mu^{-r}$$

The mixture expression is in terms of a gamma, beta and Kumer's confluent hypergeometric functions while the rth moment is in terms of gamma function. Hence

$$\begin{aligned}
E(X) &= \frac{\alpha(p+q-1)}{\mu(p-1)} \\
\text{Var}(X) &= \frac{\alpha(p+q-1)}{\mu^2 p(p-1)^2(p-2)}\begin{bmatrix} 4\alpha p^2 + 3\alpha pq - 5\alpha p - \alpha q + 2\alpha + p^2 \\ +pq - 3p + q - \alpha p^3 - \alpha p^2 q + 2 \end{bmatrix} \\
\text{Skewness} &= \frac{\alpha(p+q-1)}{\mu^3(p-1)}\begin{bmatrix} \frac{(\alpha+1)(\alpha+2)(p+q-2)(p+q-3)}{p(p-3)} \\ -\frac{3\alpha(\alpha+1)(p+q-1)(p+q-2)}{p(p-1)(p-2)} + \frac{2\alpha^2(p+q-1)^2}{(p-1)^2} \end{bmatrix}
\end{aligned}$$

6.3. Estimation

Method of Moments estimators are not accurate and solving for the inseparable parameters is difficult in gamma mixtures. Furthermore, raw moments may not exist in some of the models. The solutions to normal or score equations of ML estimation cannot, in most cases, be obtained explicitly. In the section below, we give a general formulation for the expectation-maximization (EM) algorithm for the gamma mixtures.

6.3.1. EM Algorithm

The EM algorithm was introduced by Dempster(1977) to compute maximum likelihood estimates from incomplete data. Karlis (2002) argued that the EM algorithm formulation is particularly suitable for distributions arising as mixtures since the mixing operation can be considered responsible for producing missing data.The scheme is based on bivariate distribution with two parts; the conditional distribution and mixing distribution i.e.,.

$$f(x_i, \lambda_i) = f(x_i|\lambda_i)g(\lambda_i)$$

The complete loglikelihood factorizes into two parts (see Kostas (2007)):

$$\log L = \ell = \sum_{i=1}^{n} \log f(x_i|\lambda_i) + \sum_{i=1}^{n} \log g(\lambda_i)$$

The normal or score equations are

$$\frac{\partial \ell_1}{\partial \underline{\theta}_1} = 0 \;\text{ and }\; \frac{\partial \ell_2}{\partial \underline{\theta}_2} = 0$$

The normal equations also consist of digamma [$\psi(\cdot)$] and trigamma[$\psi'(\cdot)$] functions. They also consist of a known variable X_i and its function $f(X_i|\Lambda_i)$; and the unknown variable Λ_i and its function $\phi(\Lambda_i)$, say. The functions of unobserved variables in gamma mixtures take the pattern given by:

$$\phi(\Lambda) = \Lambda,\ \Lambda^{-1},\ \text{and } \log\Lambda$$

Each of the functions in this pattern is estimated by the posterior expectation given by

$$E[\phi(\Lambda)|X] = \frac{\int_0^\infty \phi(\lambda)f(x|\lambda)g(\lambda)d\lambda}{\int_0^\infty f(x|\lambda)g(\lambda)d\lambda}$$

6.3.2. Case study: EM algorithm for Gamma I- Gamma I mixture

$$f(x|\lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}, x > 0;\ \alpha > 0,\ \lambda > 0$$

and

$$g(\lambda) = \frac{\mu^\beta}{\Gamma(\beta)}\lambda^{\beta-1}e^{-\mu\lambda}, \lambda > 0;\ \beta > 0,\ \mu > 0$$

M-Step. or conditional and mixing distributions, respectively:-

$$\ell_1 = \alpha\sum_{i=1}^n \log\lambda_i - n\log\Gamma(\alpha) + (\alpha-1)\sum_{i=1}^n \log x_i - \sum_{i=1}^n \lambda_i x_i$$
$$\ell_2 = n\beta\log\mu - n\log\Gamma(\beta) + (\beta-1)\sum_{i=1}^n \log\lambda_i - \mu\sum_{i=1}^n \lambda_i$$

$$\begin{aligned}\therefore \frac{\partial\ell_1}{\partial\alpha} &= \sum_{i=1}^n \log\lambda_i + \sum_{i=1}^n \log x_i - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\\ &= \sum_{i=1}^n \log\lambda_i + \sum_{i=1}^n \log x_i - n\psi(\alpha)\end{aligned}$$

where $\psi(\alpha)$ is a digamma function. Similarly

$$\begin{aligned}\frac{\partial\ell_2}{\partial\beta} &= n\log\mu - \sum_{i=1}^n \log\lambda_i - n\frac{\Gamma(\beta)}{\Gamma(\beta)}\\ &= n\log\mu - \sum_{i=1}^n \log\lambda_i - n\psi(\beta)\end{aligned}$$

and

$$\frac{\partial \ell_2}{\partial \mu} = \frac{n\beta}{\mu} - \sum_{i=1}^{n} \lambda_i$$

The score or normal equations are

$$\sum_{i=1}^{n} \log \lambda_i + \sum_{i=1}^{n} \log x_i - n\psi(\alpha) = 0 \tag{12}$$

$$n \log \mu + \sum_{i=1}^{n} \log \lambda_i - n\psi(\beta) = 0 \tag{13}$$

$$\frac{n\beta}{\mu} - \sum_{i=1}^{n} \lambda_i = 0 \tag{14}$$

E-step

$$\phi(\Lambda) = \log \Lambda, \quad \Lambda$$

$$E[\log \Lambda | X] = \psi(\alpha + \beta) - \log(x + \mu)$$

$$E[\Lambda | X] = \frac{\alpha + \beta}{x + \mu}$$

Let

$$s_i = \frac{\alpha + \beta}{x_i + \mu}, \quad w_i = \psi(\alpha + \beta) - \log(x_i + \mu) \tag{15}$$

The score equations now become

$$\sum_{i=1}^{n} \omega_i + \sum_{i=1}^{n} \log x_i - n\psi(\alpha) = 0 \tag{16}$$

$$n \log \mu + \sum_{i=1}^{n} \omega_i - n\psi(\beta) = 0 \tag{17}$$

$$\frac{n\beta}{\mu} - \sum_{i=1}^{n} s_i = 0 \tag{18}$$

Using Newton-Raphson iteration technique and suitable substitutions gives

$$\hat{\alpha}_{k+1} = \alpha_k - \frac{f(\alpha)}{f'(\alpha)}$$

$$\hat{\mu} = \frac{n\hat{\beta}}{\sum_{i=1}^{n} s_i} \tag{19}$$

$$\hat{\beta}_{k+1} = \beta_k - \frac{g(\beta)}{g'(\beta)}$$

Therefore the k^{th}iteration becomes

$$s_i^{(k)} = \frac{\alpha^{(k)} + \beta^{(k)}}{x_i + \mu^{(k)}}$$

and

$$\omega_i^{(k)} = \psi(\alpha^{(k)} + \beta^{(k)}) - \log(x_i + \mu^{(k)})$$

Hence the iterative scheme for parameter estimators is given by

$$\alpha^{(k+1)} = \alpha^{(k)} + \frac{n\bar{w}_i^{(k)} - n\psi(\alpha^{(k)}) + \sum_i^n \log x_i}{n\psi'(\alpha^{(k)})}$$

$$\beta^{(k+1)} = \beta^{(k)} \left\{ \frac{\begin{bmatrix} 1 - \beta^{(k)}\psi'(\beta^{(k)}) - \log \beta^{(k)} + \log \bar{s}_i^{(k)} \\ +\psi(\beta^{(k)}) + \bar{w}_i^{(k)} \end{bmatrix}}{1 - \beta^{(k)}\psi'(\beta^{(k)})} \right\}$$

$$\mu^{(k+1)} = \frac{\beta^{(k+1)}}{\bar{s}_i^{(k)}}$$

7. Conclusion

Our focus in this article is on identifying classes of mixing distributions based on Generalized Inverse Gaussian distribution. Other classes can be obtained from the beta Family which includes Beta I, Beta II and their generalizations. Classes of mixing distributions discussed can be applied to any continuous mixture. We have applied the framework on continuous gamma mixtures. In particular, we have discussed the forms gamma mixtures take and the EM algorithm for ML estimation. Extensions can be made to training of data sets to the models.

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