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Asymptotic normality of the kernel estimator of the recursive density under the censored β-mixing model

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Abstract In this paper, we establish the asymptotic normality of the recursive estimator of the density function for the right-censored random model when the data present some form of dependence. It is assumed that, the survival and the censoring times form a stationary β-mixing. Therefore this paper is part of this vast project aimed to extending the results obtained with independent variables in the dependent case.

Résumé. Dans cet article, nous établissons la normalité asymptotique de l'estimateur récursif de la fonction de densité pour le modèle aléatoire censuré à droite lorsque les données présentent une certaine forme de dépendance. On suppose que les temps de survie et de censure forment un mélange β stationnaire. Cet article fait donc partie de ce vaste projet visant à étendre les résultats obtenus avec des variables indépendantes dans le cas dépendant.

Key words: asymptotic normality; censored data; kernel density; β-mixing.
AMS 2010 Mathematics Subject Classification Objects : 62G05; 62G08; 62M10.

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1. Introduction and motivation

Density estimation is a problem of considerable interest in many applied fields such as decision theory and forecasting. Since the years 1950, Density function estimation has been studied extensively and many estimation methods have been proposed, by example the kernel smoothing method and the nearest neighbor method, see, for example,Rosenblatt, M. (1956) and Loftsgarden and Quesenberry (1965). Recently,kernel smoothing method was successfully applied in density estimation. Survival data are generally observed incompletely, due of the presence of a number of of events likely to censor the event that interests us. Another reason may be withdrawals from a clinical trial, deaths unrelated to the disease studied, etc.

In many situations, complete data $T_1, ..., T_n$ are not available because we do not have access to all the information. Let the censoring times $C_1...C_n$ be independent and identically distributed (i.i.d.) and independent of $T_i; i = 1; 2; ...; n$ of distribution function G unknown. T and C are assumed to be independent. Among the different forms of incomplete data, we find censorship and truncation. In survival analysis, censorship at right is the most frequent case of incomplete observation , and has largely been described in the literature see (Anderson, Borgan and Keiding (1993)).

Censorship is a condition in which the observation is only partially known, so we only observes the n couples (Y_i, δ_i) with $Y_i = \min(T_i, C_i)$ and $\delta_i = \mathbb{I}_{\{T_i \leq C_i\}}$,where $\mathbb{I}_A$ designates the indicator function of event A and we let $\pi_i = P[\delta_i = 1|T_i]$, this probability is called the propensity score(see Rosenbaum and Rubin (1983)). The problem posed for a long time by statisticians was that : how to find estimation and inference methods in a non-parametric setting for f and its mode θ from the n couples actually observed (Y_i, δ_i) ? In this kind of model, it is well known that the classical empirical distribution does not estimate efficiently the densities F and G. Therefore, Kaplan and Meier (1958) proposed consistent estimators F_n and G_n for F and G, respectively, by :

$$F_n(t) = \prod_{i=1}^{n} \left[1 - \frac{\delta_{(i)}}{n-i+1}\right]^{\mathbb{I}_{\{Y_{(i)} \leq t\}}} \text{ and } G_n(t) = \prod_{i=1}^{n} \left[1 - \frac{1-\delta_{(i)}}{n-i+1}\right]^{\mathbb{I}_{\{Y_{(i)} \leq t\}}}, \quad (1)$$

where $Y_{(1)}, Y_{(2)}, ..., Y_{(n)}$) denote the order statistics of $Y_1, Y_2, ..., Y_n$ and δ_i is the concomitant of $Y_{(i)}$.

The work of Kaplan and Meier (1958) presents important results regarding estimation non-parametric survival function for right-censored random variables.

At the beginning, the results establishing the properties of the proposed estimators make independence an assumption forte. However, this condition is not always admissible: in certain situations, a dependence structure is imposed on the observed data. The form of dependence considered in this article is mixing beta, a concept introduced by Volkonskii and Rozanov (1959) For any two σ-fields $\mathcal{A}$ and $\mathcal{B} \subset \mathcal{F}$ define the following measure of dependence (see Bradley (2005).):

$$\beta(\mathcal{A}, \mathcal{B}) = \sup \left\{ \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)| \right\}, \tag{2}$$

Where the latter supremum is taken over all pairs of finite partitions $(A_1, A_2, ..., A_I)$ and $(B_1, B_2, ..., B_J)$ of Ω such that $A_i \in \mathcal{A}$ for each i and $B_j \in \mathcal{B}$ for each j.
Now suppose $T := (T_k, k \in \mathbb{Z})$ is a strictly stationary sequence of random variables on $(\Omega, \mathcal{F}, \mathcal{P})$. For the given random sequence T, for any positive integer n, define the dependence coefficient as follows:

$$\beta(n) = \beta(T, n) = \sup_{j \in \mathbb{Z}} \beta(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}). \tag{3}$$

We say that the sequence (T_k) is β mixing, if the mixing coefficient $\beta(n) \to 0$ when $n \to \infty$.

As to our knowledge, there are a few papers dealing with density estimation with dependent data for the censored model.

In this paper, we consider non-parametric estimation of the density function, using a recursive kernel estimator. In this way, the estimator can be updated each time new additional observation. This recursive functionality offers many advantages : they are easy to implement and interpret and quick to calculate and does not require significant storage data. In particular cases, they also appear to be more efficient than conventional systems.

We first note that the recursive version of Rosenblatt's kernel density estimator, was introduced by Wolverton and Wolverton (1969), and was widely studied, see among many others Yamato (1971), Davies(1973). Competing recursive estimators, which may be regarded as weighted versions of estimator, were introduced and studied by **?**. Duflo (1997), Hall and Patil (1994) defined a large class of weighted recursive estimators, including all the previous recursive estimators.

The objective of the present article is to establish the asymptotic normality of the recursive estimator of the density function for the right-censored random model when the data present some form of dependence. It is assumed that survival and censored times form a stationary β mixing sequence.

It is necessary to sthochastically create an algorithm that approximates f at point t to determine the zero of the function $g : y \to f(t) - y$. According to the diagram of Robbins-Monro (see Robbins and al. (1951)), let $\hat{f}_0(t) \in \mathbb{R}$ and for all $n \geq 1$, $\hat{f}_n(t) = \hat{f}_{n-1}(t) + \gamma_n W_n(t)$ where (γ_n) is the step size. Concerning the definition of W_n at a point t, we follow Révész and Robbins. (1973), Révész (1977), (see also Tsybakov (1990), Slaoui (2019), Slaoui (2018) .

By referring to Slaoui (2019) and remplacing $\hat{\pi}_n^{-1}$ by $\bar{G}_n^{-1}$ we obtain

$$W_n = h_n^{-1}\delta_n \bar{G}_n^{-1} K(h_h^{-1}(t - Y_n)) - f_{n-1}(t), \tag{4}$$

where $\bar{G}_n = 1 - G_n$ and G_n is the Kaplan-Meier estimator associated with it, which has the expression :

$$G_n(t) = \begin{cases} \prod_{i=1}^n \left[1 - \dfrac{1-\delta_{(i)}}{n-i+1}\right]^{\mathbb{I}\{Y_{(i)} \leq t\}}, & \text{if} \quad t < Y_{(n)} \\ 0 & \text{if} \quad t \geq Y_{(n)}. \end{cases} \tag{5}$$

In this article we will work with the recursive estimator $\hat{f}_n$ of the function f defined at the point t as follows:

$$\hat{f}_n(t) = (1 - \gamma_n)\hat{f}_{n-1}(t) + \gamma_n \delta_n \bar{G}_n^{-1} K\left(h_n^{-1}(t - Y_n)\right). \tag{6}$$

we assume that $\tilde{f}_0(x) = 0$ and let $Q_n = \prod_j^n (1 - \gamma_j)$. Then in this paper we propose to study the following estimator of f at the point t : (see Slaoui (2016))

$$\hat{f}_n(t) = Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k \delta_k h_k^{-1} \bar{G}_n^{-1}(Y_k) K\left(\frac{t - Y_k}{h_k}\right). \tag{7}$$

G is known, the recursive pseudo-estimator $\tilde{f}_n$ of f estimates the common density well lifespans.

$$\tilde{f}_n(t) = Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k \delta_k h_k^{-1} \bar{G}^{-1}(Y_k) K\left(\frac{t - Y_k}{h_k}\right). \tag{8}$$

2. Hypotheses and main results

To establish our results we will need these classic hypotheses:

$\mathbf{M_1}$. $\{T_i, i \geq 1\}$ is a strictly stationary β-mixing sequence of random variable with a common distribution function (d.f),F which has a probability density function (p.d.f) f admitting finite second moments.

$\mathbf{M_2}$. The censoring times $\{C_i, 1 \leq i \leq n\}$ are i.i.d with distribution function G, and are independent of $\{T_i, i \geq 1\}$.

K. K is a Lipshitz density with true compact support satisfying $\int uK(u)du = 0$ and $\int_{\mathbb{R}} K(x)dx = 1$.

$\mathbf{D_1}$. The density $f(.)$ is twice continuously differentiable on $[0, \tau]$,where $\tau < \tau_F = \inf\{t, F(t) = 1\}$.

$\mathbf{D_2}$. The coefficient of β-mixture of T_i verifies $\beta(n) = \mathcal{O}(n^{-\nu})$ for all $\nu > 3$.

$\mathbf{D_3}$.The function f is bounded, differentiable, and $f''(t)$ is bounded on the set $\{t, f(t) > 0\}$.

$\mathbf{D_4}$. $\exists D_1 > 0$ and $\exists D_2 > 0$ such that $\sup_{u,v\in[0,\tau]} |g_{ij}(u, v)| < D_1$ and $\sup_{u\in[0,\tau]} |g(u)| < D_2$,where g_{ij} is the joint distribution of (T_i, T_j).

H. The smoothing parameter $h = h_n$ is such that $h \to 0, \gamma_n^{-1} h_n^5 \to 0$, $\xi = \lim_{n\to\infty}(n\gamma_n)^{-1}$ and $n^{-\nu} h_n Q_n^{-1} \to 0$ as $n \to \infty$.

1. $\gamma_n \in \mathcal{GS}(-\alpha)$ with $\alpha \in (1/2, 1]$.

2. $h_n \in \mathcal{GS}(-a)$ with $a \in (0, 1)$.

3. $Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k = (1 + o(1))$.

4. $Q_n \sum_{k=1}^n Q_k^{-1} \gamma_k h_k^2 = \mathcal{O}\left(h_n^2\right)$.

5. $Q_n^2 \sum_{k=1}^n Q_k^{-2} \gamma_k^2 h_k^{-1} = \mathcal{O}\left(\gamma_n h_n^{-1}\right)$.

6. Let $p := p(n)$ and $q := q(n)$ be positive integers and $k := k(n) = [n/(p + q)] \to \infty$ such that $pkQ_n^2 \gamma_n^{-1} h_n \to 1$ and $k\beta(q) \to 0$.

We recall that, a sequence (v_n) is said to have regular variation and we note $(v_n) \in \mathcal{GS}(\gamma)$ if

$$\lim_{n\to\infty} n \left[1 - \frac{v_{n-1}}{v_n}\right] = \gamma. \tag{9}$$

Remark 1. Let

$$K_j(t) = Q_j^{-1}\gamma_j\delta_j h_j^{-1}\bar{G}^{-1}(Y_j)K\left(\frac{t-Y_j}{h_j}\right),$$

and

$$\begin{aligned}
\tilde{f}_n(t) - \mathbb{E}\tilde{f}_n(t) &= Q_n\sum_{j=1}^{n} Q_j^{-1}\gamma_j\delta_j h_j^{-1}\bar{G}_j^{-1}(Y_j)K\left(\frac{t-Y_j}{h_j}\right) \\
&\quad -\mathbb{E}\left[Q_n\sum_{j=1}^{n} Q_j^{-1}\gamma_j\delta_j h_j^{-1}\bar{G}_j^{-1}(Y_j)K\left(\frac{t-Y_j}{h_j}\right)\right] \\
&= Q_n\sum_{j=1}^{n} K_j(t) - \mathbb{E}(K_j(t)) \\
&= Q_n\sum_{j=1}^{n} \tilde{K}_j(t),
\end{aligned}$$

where

$$\begin{aligned}
\tilde{K}_j(t) &= K_j(t) - \mathbb{E}(K_j(t)) \\
&= Q_j^{-1}\gamma_j\delta_j h_j^{-1}\bar{G}_j^{-1}(Y_j)K\left(\frac{t-Y_j}{h_j}\right) - \mathbb{E}\left[Q_j^{-1}\gamma_j\delta_j h_j^{-1}\bar{G}_j^{-1}(Y_j)K\left(\frac{t-Y_j}{h_j}\right)\right].
\end{aligned}$$

Then

$$\sqrt{\gamma_n^{-1}h_n}\left(\tilde{f}_n(t) - \mathbb{E}\tilde{f}_n(t)\right) = Q_n\sqrt{\gamma_n^{-1}h_n}\sum_{j=1}^{n}\tilde{K}_j(t). \tag{10}$$

Remark 2. $H.4$ shows that $p/n \to 0; (p+q)kQ_n^2\gamma_n^{-1}h_n \to 1$ and $pkQ_n^2\gamma_n^{-1}h_n \to 1$ imply that $qkQ_n^2\gamma_n^{-1}h_n \to 0$. Hence $q/p \to 0$, so that $q < p$ for large n.

In the following Theorem we will establish the asymtotic normality of $\tilde{f}$.

Theorem 1. *Under slightly restrictive hypothesis on the mixing coefficient, the kernel K, the density f, M_1, M_2, K, D_1 to D_4 and H*

$$\sqrt{\gamma_n^{-1}h_n}\left(\tilde{f}_n(t) - \mathbb{E}\tilde{f}_n(t)\right) \longrightarrow \mathcal{N}(0, \sigma^2(t)), \tag{11}$$

where $\sigma^2(t) = Q_1^{-2}\gamma_1^2\dfrac{f(t)}{h_1\bar{G}(t)}\int K^2(u)du.$

Corollary 1. *Under the hypothesis of Theorem 1, we get this follow relation*

$$\sqrt{\gamma_n^{-1}h_n}\left(\hat{f}_n(t)-f(t)\right)\longrightarrow\mathcal{N}(0,\sigma^2(t)). \tag{12}$$

For the proof of the theorem 1, we will need the following lemma

Lemma 1. *Under the hypothesis H, we have*

1. $\sqrt{\gamma_n^{-1}h_n}\left(\mathbb{E}\tilde{f}_n(t)-f(t)\right)\to 0,$
2. $\sqrt{\gamma_n^{-1}h_n}\left(\hat{f}_n(t)-\tilde{f}_n(t)\right)\to 0.$

Proof of Lemma 1. By referring to Mar and al. (2024), we had $\mathbb{E}\tilde{f}_n(t)-f(t)=\mathcal{O}(h_n^2)$, then

$$\begin{aligned}\sqrt{\gamma_n^{-1}h_n}\left(\mathbb{E}\tilde{f}_n(t)-f(t)\right) &= \sqrt{\gamma_n^{-1}h_n}\left(\mathcal{O}(h_n^2)\right)\\ &= \mathcal{O}(\gamma_n^{-1}h_n^5)\\ &\to 0.\end{aligned}$$

Moreover

$$\begin{aligned}|\hat{f}_n(t)-\tilde{f}_n(t)| &= \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}}h_n^2\right)\\ &\Rightarrow \sqrt{\gamma_n^{-1}h_n}\left(\hat{f}_n(t)-\tilde{f}_n(t)\right)\\ &\leq C\sqrt{\frac{\gamma_n^{-1}h_n^5\log\log n}{n}}\\ &\to 0. \blacksquare\end{aligned}$$

Remark 3. Given that

$$\begin{aligned}\sqrt{\gamma_n^{-1}h_n}\left(\hat{f}_n(t)-f(t)\right) &= \sqrt{\gamma_n^{-1}h_n}\left(\hat{f}_n(t)-\tilde{f}_n(t)\right)+\sqrt{\gamma_n^{-1}h_n}\left(\tilde{f}_n(t)-\mathbb{E}\tilde{f}_n(t)\right)\\ &+ \sqrt{\gamma_n^{-1}h_n}\left(\mathbb{E}\tilde{f}_n(t)-f(t)\right),\end{aligned}$$

then

$$\begin{aligned}\sqrt{\gamma_n^{-1}h_n}\left(\hat{f}_n(t)-f(t)\right)\sim\sqrt{\gamma_n^{-1}h_n}\left(\tilde{f}_n(t)-\mathbb{E}\tilde{f}_n(t)\right) &= Q_n\sqrt{\gamma_n^{-1}h_n}\sum_{j=1}^{n}\tilde{K}_j(t)\\ &= \mathcal{L}_n(t).\end{aligned}$$

Proof of Theorem 1. In order to establish the asymptotic normality of the recursive density estimators proposed under a strong mixture of random variables, we use the sectioning device introduced by Doob (1953). Let's divide this sum $\sum_{i=1}^{n} \tilde{K}j(t)$ into large and small blocks as follows:

For $\{m = 1, 2, \ldots, k\}$ we divide $\{1, 2, \ldots, n\}$ into k large p bloks I_m and k small blocks q bloks J_m where :
$I_m = \{i, i = (m-1)(p+q)+1, \ldots, (m-1)(p+q)+p\}$ contains p elements for each $m = 1, 2, \ldots, k$.
$J_m = \{j, (m-1)(p+q)+p+1, \ldots, m(p+q)\}$ contains q elements for each $m = 1, 2, \ldots, k$. The remaining points will constitute the whole $\{l; k(p+q)+1 \leq l \leq n\}$ which can be empty.

For $m = 1, 2, \ldots, k$, let's define the following sums :

$$S_{n1} = \sum_{j=(m-1)(p+q)+1}^{(m-1)(p+q)+p} \tilde{K}_j, \; V_m = \sum_{j=(m-1)(p+q)+p+1}^{m(p+q)} \tilde{K}_j(t), \; S_{n3} = \sum_{l=k(p+q)+1}^{n} \tilde{K}_l. \tag{13}$$

Then

$$S_n = \sum_{j=1}^{n} \tilde{K}_j(t) = \sum_{j=(m-1)(p+q)+1}^{(m-1)(p+q)+p} \tilde{K}_j + \sum_{m=1}^{k} V_m + \sum_{l=k(p+q)+1}^{n} \tilde{K}_l = S_{n1} + S_{n2} + S_{n3}.$$

Therefore

$$\mathcal{L}_n(t) = Q_n \sqrt{\gamma_n^{-1} h_n} \left\{ S_{n1} + S_{n2} + S_{n3} \right\}.$$

To establish the Theorem 1, we then need to prove that $\mathcal{L}_n(t) \longrightarrow N(0, \sigma^2(t))$ and for this we will first establish that

$$Q_n^2 \gamma_n^{-1} h_n \left[\mathbb{E}(S_{n2}^2) + \mathbb{E}(S_{n3}^2) \right] \longrightarrow 0$$

and that

$$Q_n \sqrt{\gamma_n^{-1} h_n} S_{n1} \longrightarrow N(0, \sigma^2(t))$$

Let $k_m = (m-1)(p+q)$ and $k_m^* = (m-1)(p+q)+p$. We have

$$\begin{aligned}
S_{n1} &= \sum_{m=1}^{k} y_{mn} \qquad with \qquad y_{mn} = \sum_{i=k_m+1}^{k_m+p} \tilde{K}_i, \\
S_{n2} &= \sum_{m=1}^{k} \sum_{i=k_m^*+1}^{k_m^*+q} \tilde{K}_i,
\end{aligned}$$

,

$$\begin{aligned}
Q_n^2\gamma_n^{-1}h_n\mathbb{E}(S_{n2}^2) &= kqQ_n^2\gamma_n^{-1}h_n Var(\tilde{K}_1) + 2Q_n^2\gamma_n^{-1}h_n \sum_{m=1}^{k} \sum_{k_m^*+1\le j\le k_m^*+q} cov(\tilde{K}_i, \tilde{K}_j) \\
&+ 2Q_n^2\gamma_n^{-1}h_n \sum_{1\le i<j\le k} cov\left(\sum_{i=k_m^*+1}^{k_m^*+q} \tilde{K}_i, \sum_{j=k_m^*+1}^{k_m^*+q} \tilde{K}_j\right) \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

Now, let's prove that I_1, I_2 and I_3 tend to 0. We have

$$\begin{aligned}
Var(\tilde{K}_1) &= Var\left(K_1(t) - \mathbb{E}K_1(t)\right) \\
&= \mathbb{E}\left(Q_1^{-2}\gamma_1^2\delta_1h_1^{-2}\bar{G}^{-2}(Y_1)K^2\left(\frac{t-Y_1}{h_1}\right)\right) - \left[\mathbb{E}Q_1^{-1}\gamma_1\delta_1h_1^{-1}\bar{G}_1^{-1}(Y_1)K\left(\frac{t-Y_1}{h_1}\right)\right]^2 \\
&= Q_1^{-2}\gamma_1^2h_1^{-2}\mathbb{E}\left(\delta_1\bar{G}^{-2}(Y_1)K^2\left(\frac{t-Y_1}{h_1}\right)\right) - Q_1^{-2}\gamma_1^2h_1^{-2}\left[\mathbb{E}\delta_1\bar{G}_1^{-1}(Y_1)K\left(\frac{t-Y_1}{h_1}\right)\right]^2 \\
&= Q_1^{-2}\gamma_1^2h_1^{-2}\mathbb{E}\left[\mathbb{E}\left(\bar{G}^{-2}\mathbb{1}_{T_1\le C_1}(Y_1)K^2\left(\frac{t-Y_1}{h_1}\right)\right)|T_1\right] \\
&- Q_1^{-2}\gamma_1^2h_1^{-2}\left[\mathbb{E}\left[\mathbb{E}\mathbb{1}_{T_1\le C_1}\bar{G}_1^{-1}(Y_1)K\left(\frac{t-Y_1}{h_1}\right)\right]^2|T_1\right]^2 \\
&= Q_1^{-2}\gamma_1^2h_1^{-2}\mathbb{E}\left(K^2\left(\frac{t-T_1}{h_1}\right)\frac{1}{\bar{G}(T)}\right) - Q_1^{-2}\gamma_1^2h_1^{-2}\mathbb{E}^2K\left(\frac{t-T_1}{h_1}\right) \\
&= Q_1^{-2}\gamma_1^2\left[h_1^{-2}\mathbb{E}\left(K^2\left(\frac{t-T_1}{h_1}\right)\frac{1}{\bar{G}(T)}\right) - h_1^{-2}\mathbb{E}^2K\left(\frac{t-T_1}{h_1}\right)\right] \\
&= Q_1^{-2}\gamma_1^2\left[h_1^{-1}\int K^2(u)\frac{f(t-hu)}{\bar{G}(t-hu)}du - \left(\int K\left(t-hu\right)du\right)^2\right] \\
&= Q_1^{-2}\gamma_1^2\, V
\end{aligned}$$

where $V = h_1^{-1}\int K^2(u)\dfrac{f(t-hu)}{\bar{G}(t-hu)}du - \left(\int K\left(t-hu\right)du\right)^2$
Furthermore, by the Dominated Convergence Theorem with a development of Taylor and the hypothesis D_1, we obtain

$$V \to \frac{f(t)}{h_1 \bar{G}(t)} \int K^2(u)du$$

$$Var(\tilde{K}_1) \to Q_1^{-2}\gamma_1^2 \frac{f(t)}{h_1 \bar{G}(t)} \int K^2(u)du = \sigma^2(t). \tag{14}$$

Combining (14) and the Remark 2, it follows that $I_1 = 2kq\sigma^2(t)Q_n^2\gamma_n^{-1}h_n \to 0$ and then

$$|I_2| \le 2Q_n^2\gamma_n^{-1}h_n \sum_{m=1}^{k} \sum_{k_m^*+1\le j\le k_m^*+q} |cov(\tilde{K}_i, \tilde{K}_j)| \le 2Q_n^2\gamma_n^{-1}h_n \sum_{1\le i\le j\le n} |cov(\tilde{K}_i, \tilde{K}_j)|$$

Let $C = Q_i^{-1}\gamma_i Q_j^{-1}\gamma_j$, we have

$$\begin{aligned}
|cov(\tilde{K}_i, \tilde{K}_j)| &= |Cov\left(K_i(t) - \mathbb{E}K_i(t)\right), \left(K_j(t) - \mathbb{E}K_j(t)\right)| \\
&= |Cov\left(K_i(t), K_j(t)\right)| \\
&= \left| Cov\left(Q_i^{-1}\gamma_i h_i^{-1}\delta_i \bar{G}_i^{-1}(Y_i) K\left(\frac{t-Y_i}{h_i}\right), Q_j^{-1}\gamma_j \delta_j \bar{G}_j^{-1}(Y_j) K\left(\frac{t-Y_j}{h_j}\right)\right)\right| \\
&\le Q_i^{-1}\gamma_i Q_j^{-1}\gamma_j h_i^{-1}h_j^{-1}\mathbb{E}\left[K\left(\frac{t-Y_i}{h_i}\right)K\left(\frac{t-Y_j}{h_j}\right)\delta_i\bar{G}_i^{-1}(Y_i)\delta_j\bar{G}_j^{-1}(Y_j)\right] \\
&+ Q_i^{-1}\gamma_i Q_j^{-1}\gamma_j h_i^{-1}h_j^{-1}\mathbb{E}\left[K\left(\frac{t-Y_i}{h_i}\right)\delta_i\bar{G}_i^{-1}(Y_i)\right]\mathbb{E}\left[K\left(\frac{t-Y_j}{h_j}\right)\delta_j\bar{G}_j^{-1}(Y_j)\right] \\
&\le \frac{Ch_i^{-1}h_j^{-1}}{\bar{G}^2(\tau)}\mathbb{E}\left[K\left(\frac{t-Y_i}{h_i}\right)K\left(\frac{t-Y_j}{h_j}\right)\right] \\
&+ Ch_i^{-1}h_j^{-1}\mathbb{E}\left[K\left(\frac{t-Y_i}{h_i}\right)\bar{G}_i^{-1}(Y_i)\right]\mathbb{E}\left[K\left(\frac{t-Y_j}{h_j}\right)\bar{G}_j^{-1}(Y_j)\right] \\
&\le \frac{C}{\bar{G}^2(\tau)}\int\int K(u)K(z)\times\left[g_{ij}(t-uh_i, t-zh_j) + g_i(t-uh_i)g_j(t-zh_j)\right]dudz \\
&= \mathcal{O}(Q_i^{-1}\gamma_i Q_j^{-1}\gamma_j).
\end{aligned}$$

Then

$$|cov(\tilde{K}_i, \tilde{K}_j)| = \mathcal{O}(Q_i^{-1}\gamma_i Q_j^{-1}\gamma_j). \tag{15}$$

Next, to evaluate the asymptotic behavior of I_2, we define the sets

$$A_1 = \{(i,j) \text{ such that } 1 \le |i-j| \le t_n\}$$

and

$$A_2 = \{(i,j) \text{ such that } t_n + 1 \leq |i-j| \leq n-1\}$$

where $t_n = o(n)$. Let

$$\mathcal{T}_{1,n} = 2Q_n^2 \gamma_n^{-1} h_n \sum_{(i,j)\in A_1} |cov(\tilde{K}_i, \tilde{K}_j)|,$$

and

$$\mathcal{T}_{2,n} = 2Q_n^2 \gamma_n^{-1} h_n \sum_{(i,j)\in A_2} |cov(\tilde{K}_i, \tilde{K}_j)|.$$

Then by the relation (15), the hypothesis H and applying lemma A.2 in Slaoui (2019), we have

$$\begin{aligned}
|\mathcal{T}_{1,n}| &\leq 2MQ_n^2 \gamma_n^{-1} h_n \sum_{(i,j)\in A_1} Q_i^{-1}\gamma_i Q_j^{-1}\gamma_j \\
&= \mathcal{O}(t_n h_n).
\end{aligned}$$

For A_2 we use an extension of Davydov's inequality for mixing processes (see Rio (1951))and taking into account of the inequality.

$$\alpha_n \leq 2^{-1}\beta_n \leq \beta_n. \tag{16}$$

This leads, for all $i \neq j$, to

$$|cov(\tilde{K}_i, \tilde{K}_j)| \leq 2^{-1} c\beta(|i-j|)$$

Therefore, using (D_2), we have

$$\begin{aligned}
\mathcal{T}_{2,n} &\leq 2cQ_n^2 \gamma_n^{-1} h_n \sum_{j=1}^{n} \sum_{t_n+1\leq k\leq n-1} 2^{-1}\beta_k \\
&< cnQ_n^2 \gamma_n^{-1} h_n \int_{t_n+1}^{n-1} k^{-\nu} dk \\
&= \mathcal{O}(nQ_n^2 t_n^{1-\nu} \gamma_n^{-1} h_n).
\end{aligned}$$

Choosing $t_n = n^{\frac{1}{\nu}} \gamma_n^{\frac{-1}{\nu}} Q_n^{\frac{2}{\nu}} h_n^{-1}$ and under **H** we get

$$I_2 = \mathcal{T}_{1,n} + \mathcal{T}_{2,n} = \mathcal{O}(n^{\frac{1}{\nu}} \gamma_n^{\frac{-1}{\nu}} Q_n^{\frac{2}{\nu}}) = o(1)$$

Thus

$$I_2 \to 0. \tag{17}$$

Similary,

$$\begin{aligned}
|I_3| &= 2Q_n^2\gamma_n^{-1}h_n \sum_{1\le i<j<k} cov\left(\sum_{j=k_i^*+1}^{k_i^*+q} \tilde{K}_j, \sum_{i=k_j^*+1}^{k_j^*+q} \tilde{K}_i\right) \\
&\le 2Q_n^2\gamma_n^{-1}h_n \sum_{1\le i<j<k} |cov\left(\tilde{K}_i, \tilde{K}_j\right)| \\
&\to 0.
\end{aligned}$$

Finally, from the fact that $n - k(p+q) < p+q < 2p$ and the remark 2 we obtain that

$$\begin{aligned}
Q_n^2\gamma_n^{-1}h_n\mathbb{E}(S_{n3}^2) &= Q_n^2\gamma_n^{-1}h_n(n-k(p+q))Var(\tilde{K}_1) \\
&+ 2Q_n^2\gamma_n^{-1}h_n \sum_{k(p+q)+1\le i<j\le n} cov\left(\tilde{K}_i, \tilde{K}_j\right) \\
&\le CpQ_n^2\gamma_n^{-1}h_n + 2Q_n^2\gamma_n^{-1}h_n \sum_{1\le i<j\le n} cov\left(\tilde{K}_i, \tilde{K}_j\right) \\
&\to 0.
\end{aligned}$$

Thus

$$Q_n^2\gamma_n^{-1}h_n\left[\mathbb{E}(S_{n2}^2) + \mathbb{E}(S_{n3}^2)\right] \to 0$$

and it remains to show that

$$Q_n\sqrt{\gamma_n^{-1}h_n}S_{n1} \longrightarrow N(0, \sigma^2(t))$$

by, first, establishing

$$Var\left(Q_n\sqrt{\gamma_n^{-1}h_n}S_{n1}\right) \to \sigma^2(t) \tag{18}$$

$$\left|\mathbb{E}\left[exp\left(it\beta_n S_{n1}\right)\right] - \prod_{m=1}^{k} \mathbb{E}\left\{exp(it\beta_n y_{mn})\right\}\right| \to 0 \tag{19}$$

$$\beta_n^2 \sum_{m=1}^{k} \mathbb{E}[y_{mn}^2 \mathbb{I}_{\{y_{mn} > \varepsilon \beta_n^{-1} \sigma(t)\}}] \longrightarrow 0 \tag{20}$$

We first prove (18).

$$\begin{aligned} Var\left(Q_n \sqrt{\gamma_n^{-1} h_n} S_{n1}\right) &= pkQ_n^2 \gamma_n^{-1} h_n Var(\tilde{K}_1) \\ &+ 2Q_n^2 \gamma_n^{-1} h_n \sum_{m=1}^{k} \sum_{k_m+1 \le i < j \le k_m+p} cov(\tilde{K}_i, \tilde{K}_j) \\ &+ 2Q_n^2 \gamma_n^{-1} h_n \sum_{1 \le i < j < k} cov\left(\sum_{j=k_i+1}^{k_i+p} \tilde{K}_j, \sum_{i=k_j+1}^{k_j+p} \tilde{K}_i\right) \end{aligned}$$

From the proof of (17) we get

$$\left\{ (2Q_n^2 \gamma_n^{-1} h_n \sum_{m=1}^{k} \sum_{k_m+1 \le i < j \le k_m+p} cov(\tilde{K}_i, \tilde{K}_j) \right.$$
$$\left. + Q_n^2 \gamma_n^{-1} h_n \sum_{1 \le i < j < k} cov\left(\sum_{j=k_i+1}^{k_i+p} \tilde{K}_j, \sum_{i=k_j+1}^{k_j+p} \tilde{K}_i\right) \right\} \to 0$$

The remark 2 and (14) together imply that

$$pkQ_n^2 \gamma_n^{-1} h_n Var(\tilde{K}_1) \to \sigma^2(t).$$

As to (19), by using the application of Volkonski and Rozanov's (1959) inequality and (16), we obtain

$$\begin{aligned}
&\left|\mathbb{E}\left[\exp\left(it\beta_n\sum_{m=1}^{k} y_{mn}\right)\right] - \prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\beta_n y_{mn})\right\}\right| \\
&\le \left|\mathbb{E}\left[\exp\left(it\alpha_n\sum_{m=1}^{k} y_{mn}\right)\right] - \mathbb{E}\left[\exp\left(it\beta_n\sum_{m=1}^{k} y_{mn}\right)\right]\right| \\
&\quad + \left|\mathbb{E}\left[\exp\left(it\alpha_n\sum_{m=1}^{k} y_{mn}\right)\right] - \prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\beta_n y_{mn})\right\}\right| \\
&\le \left|\mathbb{E}\left[\exp\left(it\alpha_n\sum_{m=1}^{k} y_{mn}\right)\right] - \prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\beta_n y_{mn})\right\}\right| \\
&\le \left|\mathbb{E}\left[\exp\left(it\alpha_n\sum_{m=1}^{k} y_{mn}\right)\right] - \prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\alpha_n y_{mn})\right\}\right| \\
&\le 8(k-1)\beta(n) \to 0.
\end{aligned}$$

Indeed

$$\mathbb{E}\left[\exp\left(it\alpha_n\sum_{m=1}^{k} y_{mn}\right)\right] \le \mathbb{E}\left[\exp\left(it\alpha_n\sum_{m=1}^{k} y_{mn}\right)\right], \tag{21}$$

and

$$\begin{aligned}
\prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\alpha_n y_{mn})\right\} &\le \prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\alpha_n\sum_{m=1}^{k} y_{mn})\right\} \\
&\le \prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\beta_n\sum_{m=1}^{k} y_{mn})\right) .
\end{aligned} \tag{22}$$

So it comes from (21) and (22) that

$$\begin{aligned}
&\left|\mathbb{E}\left[exp\left(it\alpha_n\sum_{m=1}^{k} y_{mn}\right)\right] - \prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\beta_n y_{mn})\right\}\right| \\
&\le \left|\mathbb{E}\left[exp\left(it\alpha_n\sum_{m=1}^{k} y_{mn}\right)\right] - \prod_{m=1}^{k}\mathbb{E}\left\{\exp(it\alpha_n y_{mn})\right\}\right| .
\end{aligned}$$

By using H_3, we have

$$\begin{aligned}
|y_{nm}| &= \left|\sum_{l=k_m+1}^{k_m+p} \tilde{K}_l\right| \leq \sum_{l=k_m+1}^{k_m+p} |\tilde{K}_l| \\
&\leq \sum_{l=k_m+1}^{k_m+p} |K_l| = \sum_{l=k_m+1}^{k_m+p} Q_l^{-1}\gamma_l\delta_l h_l^{-1}\bar{G}_l^{-1}(Y_l)K\left(\frac{t-Y_l}{h_l}\right) \\
&\leq \frac{||K||_\infty}{\bar{G}(\tau)} \sum_{l=k_m+1}^{k_m+p} Q_l^{-1}\gamma_l h_l^{-1} \\
&\leq Q_n^{-1}\frac{||K||_\infty}{\bar{G}(\tau)} Q_n \sum_{l=1}^{n} Q_l^{-1}\gamma_l h_l^2 \\
&\leq C\frac{||K||_\infty h_n}{Q_n\bar{G}(\tau)}.
\end{aligned}$$

Which implies

$$|y_{nm}| \leq C\frac{||K||_\infty h_n}{Q_n\bar{G}(\tau)}. \tag{23}$$

To prove (20) we use (23) and the assumptions $\mathbf{K_1}$, $\mathbf{D_3}$ and $\mathbf{H}$. So

$$\beta_n|y_{mn}| \leq Ckn^{-\nu}h_n\frac{||K||_\infty}{Q_n\bar{G}(\tau)} \to 0.$$

Then, we have for n large enough, the set $\left\{y_{mn} > \varepsilon\beta_n^{-1}\sigma(t)\right\}$ becomes empty set. This completes the proof of Theorem 1. ■

Proof of Corollary 1. By using this decomposition,

$$\hat{f}_n(t) - f(t) = \left(\mathbb{E}\tilde{f}_n(t) - f(t)\right) + \left(\hat{f}_n(t) - \tilde{f}_n(t)\right) + \left(\tilde{f}_n(t) - \mathbb{E}\tilde{f}_n(t)\right),$$

we have

$$\begin{aligned}
\sqrt{\gamma_n^{-1}h_n}(\hat{f}_n(t) - f(t)) &= \sqrt{\gamma_n^{-1}h_n}\left(\mathbb{E}\tilde{f}_n(t) - f(t)\right) \\
&+ \sqrt{\gamma_n^{-1}h_n}\left(\hat{f}_n(t) - \tilde{f}_n(t)\right) + \sqrt{\gamma_n^{-1}h_n}\left(\tilde{f}_n(t) - \mathbb{E}\tilde{f}_n(t)\right) \\
&:= T_1 + T_2 + T_3
\end{aligned}$$

where T_1 and T_2 are negligible. Indeed, under the hypothesis K, D_1 and using the techniques of calculating expectation by conditioning, and a Taylor expansion to order 2, it follows that

$$\left|\mathbb{E}\tilde{f}_n(t) - f(t)\right| = \mathcal{O}(h_n^2)$$

and therefore under the condition H we have this follow equality

$$\sqrt{\gamma_n^{-1}h_n}\left|\mathbb{E}\tilde{f}_n(t) - f(t)\right| = \mathcal{O}\left(\sqrt{\gamma_n^{-1}h_n^5}\right) = o(1).$$

Similarly, under the conditions M_1 and M_2 and using the Law of Iterated Logarithm (LIL) for right-censored data (see Deheuvels and Einmahl (2000)) and a change of variable we obtain (see Mar and al. (2024))

$$|\hat{f}_n(t) - \tilde{f}_n(t)| = \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}}h_n^2\right)$$

then under the hypothesis H, we get

$$\sqrt{\gamma_n^{-1}h_n}|\hat{f}_n(t) - \tilde{f}_n(t)| = \mathcal{O}\left(\sqrt{\frac{\log\log n}{n}\gamma_n^{-1}h_n^5}\right) = o(1)$$

The proof is completed now. ■

3. Conclusion

This article highlights specific results concerning the censored observations at on the right, which is the most common type of censorship. We established the asymptotic normality of a recursive estimator with right-censored data in the case β-mixing. we postpone the study of the following points for future research:

(i) It would be relevant to examine the extension of our results in the case of real data and a simulation in the same cases of dependence.

(i) The asymptotic normality of this same estimator with other forms of dependencies (α-mixing, ϕ-mixing...) would be a path to follow .

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