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The GAR of the unified welfare Foster-Thorbecke-Greer-Sen-Kakwani index

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Abstract In this paper, we use a direct and specific way of finding a General Asymptotic Representation of Kakwani welfare index. The Kakwani measure includes the Sen index and the Foster-Thorbecke-Geer index. A GAR for Kakwani has been derived in Lo (2013) as a particular case of the GAR of the General Poverty index. See Page 3624 for a full abstract.

Key words: functional empirical function; functional empirical process; index asymptotic representation; Kakwani poverty measure; Functional Brownian Bridge.
AMS 2010 Mathematics Subject Classification Objects : 62E20, 62F05

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Full Abstract. In this paper, we use a direct and specific way of finding a General Asymptotic Representation of Kakwani welfare index. The Kakwani measure includes the Sen index and the Foster-Thorbecke-Geer index. A GAR for Kakwani has been derived in Lo (2013) as a particular case of the GAR of the General Poverty index. However that general approach uses highly mathematical tools. But that approach does not use the *fep-lrfep* method as exposed in Lo et al. (2023). So, a direct treatment based on the new unified formulation is to be considered and that would serve as a validation of the first result.

Résumé. (French abstract). Dans cet article, nous utilisons une méthode directe et spécifique pour trouver une représentation asymptotique générale de l'indice de bien-être de Kakwani. La mesure de Kakwani comprend l'indice de Sen et l'indice de Foster-Thorbecke-Geer. Un GAR pour Kakwani a été dérivé dans Lo (2013) comme cas particulier du GAR de l'indice général de pauvreté. Cependant, cette approche générale utilise des outils hautement mathématiques mais n'utilise pas la méthode *fep-lrfep* telle qu'exposée dans Lo et al. (2023). Ainsi, un traitement direct basé sur la nouvelle formulation unifiée doit être envisagé et cela servirait de validation du premier résultat.

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1. Introduction

Estimating parameters and studying the asymptotic behavior of the estimators are common practices in Statistics. In that sense, approaches based on the functional empirical processes (*fep*) can be applied to a number of statistics and indexes, that are used in Mathematical Statistics and in many other disciplines.

In this paper, we use a direct and specific way of finding a General Asymptotic Representation of Kakwani welfare index. The layout of the paper is as follows. We describe the statistic under study and give some reminders on empirical functions

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in the next two subsections, then the results on the GAR are given in Section 2. The proofs of all the results are given in Section 3 and we conclude the paper in Section 4.

1.1. Description of the statistic under study

Let X_1, X_2, etc. be a series of independent observations of the income variable of some population, with underlying distribution F. For each n fixed, we consider the order statistic $X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n}$ of the sample $(X_1, \cdots, X_n)$. We are given a poverty line Z which is set as follows. An individual is considered as poor if and only if his income is below Z. That poverty line is supposed be such that $F(Z) < 1$, otherwise all the population are supposed to be poor persons, which is not relevant in a non-trivial study. Let

$$Q_n = \sum_{j=1}^{n} 1_{(X_j \leq Z)}$$

be the empirical number of poor individual in the population. The Kakwani poverty measure of parameter $s > 0$ is defined as

$$P_{KAK,n}(s) = \frac{Q_n}{n\Phi_s(Q_n)} \sum_{j=1}^{n} (Q_n - j + 1)^s \left(\frac{Z - X_{j,n}}{Z}\right),$$

where

$$\Phi_s(Q_n) = \sum_{j=1}^{Q_n} j^s$$

This statistic is one of the most important tools for monitoring poverty in Economics and is a generalization of the important Sen statistics which is

$$Sen(n) = P_{KAK,n}(1) = \frac{2}{n(Q_n + 1)} \sum_{j=1}^{n} (Q_n - j + 1) \left(\frac{Z - X_{j,n}}{Z}\right).$$

Our objective is to obtain the General Asymptotic Representation *GAR* using the functional empirical process (*fep*) and the Lo residual empirical process (*lrfep*) in the spirit of Lo et al. (2023). Before we go further, we need some recalls for stating this objective.

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1.2. Recalls on the empirical functions

In this part, we complete the notation we already gave and specify our probability space.

The notation below concerns the univariate random distributions. We are going to describe the general Gaussian field in which we present our results. Indeed, we use a unified approach when dealing with the asymptotic theories of the welfare statistics. It is based on the Functional Empirical Process (*fep*) and its Functional Brownian Bridge (*fbb*) limit. It is laid out as follows.

When we deal with the asymptotic properties of one statistic or index at a fixed time, we suppose that we have a non-negative random variable of interest which may be the income or the expense X whose probability law on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, the Borel measurable space on $\mathbb{R}$, is denoted by $\mathbb{P}_X$. We consider the space $\mathcal{F}$ of measurable real-valued functions f defined on $\mathbb{R}$ such that

$$V_X(f) = \int (f - \mathbb{E}_X(f))^2 d\mathbb{P}_X = \mathbb{E}(f(X) - \mathbb{E}(f(X))^2 < +\infty,$$

where

$$\mathbb{E}_X(f) = \mathbb{E}f(X).$$

On this functional space $\mathcal{F}$, which is endowed with the L_2-norm

$$\|f\|_2 = \left(\int f^2 d\mathbb{P}_X\right)^{1/2},$$

we define the Gaussian process $\{\mathbb{G}(f), f \in \mathcal{F}\}$, which is characterized by its variance-covariance function

$$\Gamma(f,g) = \int^2 (f - \mathbb{E}_X(f))(g - \mathbb{E}_X(g))d\mathbb{P}_X, (f,g) \in \mathcal{F}^2. \tag{1}$$

This Gaussian process is the asymptotic weak limit of the sequence of functional empirical processes (fep) defined as follows. Let $X_1, X_2, ...$ be a sequence of independent copies of X. For each $n \geq 1$, we define the functional empirical process associated with X by

$$\mathbb{G}_n(f) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f(X_j) - \mathbb{E}f(X_j)), f \in \mathcal{F},$$

and denote the integration with respect to the empirical measure by

$$\mathbb{P}_n(f) = \frac{1}{n} \sum_{i=1}^{n} f(X_i), \ f \in \mathcal{F}.$$

Let us denote by $\ell^\infty(T)$ the space of real-valued bounded functions defined on $T = \mathbb{R}$ equipped with its uniform topology. In the terminology of the weak

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convergence theory, the sequence of objects $\mathbb{G}_n$ weakly converges to $\mathbb{G}$ in $\ell^\infty(\mathbb{R})$, as stochastic processes indexed by $\mathcal{F}$, whenever it is a Donsker class. The details of this highly elaborated theory may be found in Billingsley (1968), Pollard (1984), van der Vaart and Wellner (1996) and similar sources.

Here, we only need the convergence in finite distributions which is a simple consequence of the multivariate central limit theorem, as described in Chapter 3 in Lo *et al.* (2016).

We will use the Rényi's representation of the random variable X_i's of interest by means of the (*cdf*) F of X as follows

$$X =_d F^{-1}(U),$$

where U is a uniform random variable on $(0,1)$, $=_d$ stands for the equality in distribution and F^{-1} is the generalized inverse of F, defined by

$$F^{-1}(s) = \inf\{x, F(x) \geq s\},\ s \in (0,1).$$

Based on these representations, we may and do assume that we are on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ holding a sequence of independent $(0,1)$-uniform random variables U_1, U_2, ..., and the sequence of independent observations of X are given by

$$X_1 = F^{-1}(U_1),\quad X_2 = F^{-1}(U_2),\quad etc. \tag{2}$$

For each $n \geq 1$, the order statistics of $U_1, ..., U_n$ and of $X_1, ..., X_n$ are denoted respectively by $0 \equiv U_{0,n} < U_{1,n} \leq \cdots \leq U_{n,n} < U_{n+1,n} \equiv 1$ and $-\inf \equiv X_{0,n} < X_{1,n} \leq \cdots \leq X_{n,n} < X_{n+1,n}$ ($0 = -\inf \equiv X_{0,n}$ for a non-negative random variable X).

We also associate the sequence of $(U_n)_{n\geq 1}$ with the sequence of real-argument empirical functions

$$\begin{aligned}
\mathbb{U}_n(s) &= \frac{1}{n}\#\{j, 1 \leq j \leq n,\ U_j \leq s\},\ s \in (0,1)\ n \geq 1 && (3)\\
&= \sum_{j=1}^{n} \frac{j}{n} 1_{(U_{j,n} \leq s < U_{j+1,n})}, && (4)
\end{aligned}$$

and the sequence of real-argument uniform quantile function

$$\mathbb{V}_n(s) = U_{1,n} 1_{(s=0)} + \sum_{j=1}^{n} U_{j,n} 1_{((j-1)/n < s \leq (j/n))},\ s \in (0,1),\ n \geq 1 \tag{5}$$

and next, the sequence of real-argument uniform empirical processes

$$\alpha_n(s) = \sqrt{n}(\mathbb{U}_n - s), \tag{6}$$

for $s \in (0,1)$ and ≥ 1, and the sequence of real-argument uniform quantile processes

$$\gamma_n(s) = \sqrt{n}(s - \mathbb{V}_n), \ s \in (0,1) \ n \geq 1. \tag{7}$$

The same can be done for the sequence $(X_n)_{n\geq 1}$, and we obtain the associated sequence of real empirical processes as

$$\mathbb{G}_{n,r}(x) = \sqrt{n}\left(\mathbb{F}_n(x) - F(x)\right), \ x \in \mathbb{R}, \ n \geq 1, \tag{8}$$

where

$$\mathbb{F}_n(x) = \frac{1}{n}\#\{j, 1 \leq j \leq n, \ X_j \leq x\}, \ x \in \mathbb{R} \ n \geq 1 \tag{9}$$

is the associated sequence of empirical functions. We also have the associated sequence of quantile processes

$$\mathbb{Q}_n(x) = \sqrt{n}\left(\mathbb{F}_{(n)}^{-1}(s) - F^{-1}(s)\right), \ s \in (0,1), \ n \geq 1 \tag{10}$$

where, for $n \geq 1$,

$$\mathbb{F}_n^{-1}(s) = X_{1,n}1_{(0 \leq s \leq 1/n)} + \sum_{j=1}^{n} X_{j,n}1_{((j-1)/n \leq s \leq (j/n))}, \ s \in (0,1), \tag{11}$$

is the associated sequence of quantile processes.

By passing, we recall that $\mathbb{F}_n^{-1}$ is actually the generalized inverse of $\mathbb{F}_{(n)}$ and for the uniform sequence, we have

$$\mathbb{V}_n = \mathbb{U}_n^{-1}. \tag{12}$$

In virtue of Representation (2), we have the following remarkable relations

$$\mathbb{G}_{n,r}(x) = \alpha_n(F(x)), \ x \in \mathbb{R} \tag{13}$$

and

$$\mathbb{Q}_n(x) = \sqrt{n}\left(F^{-1}(\mathbb{V}_n(s)) - F^{-1}(s)\right) \ s \in (0,1), \ n \geq 1. \tag{14}$$

We also have the following relations between the empirical functions and quantile functions

$$\mathbb{F}_n(x) = \mathbb{U}_n(F(x)), \ x \in \mathbb{R} \tag{15}$$

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and

$$\mathbb{F}_n^{-1}(s) = F^{-1}(\mathbb{V}_{(n)}(s)),\ s \in (0,1),\ n \geq 1. \tag{16}$$

As well, the real-argument and function-argument empirical processes are related as follows: for $n \geq 1$,

$$\mathbb{G}_{n,r}(x) = \mathbb{G}_n(f_x^*),\ \alpha_n(s) = \mathbb{G}_n(\tilde{f}_s),\ s \in (0,1),\ x \in \mathbb{R}, \tag{17}$$

where for any $x \in \mathbb{R}$, $f_x^* = 1_{]-\infty,x]}$ is the indicator function of $]-\infty,x]$ and for $s \in (0,1)$, $f_s = 1_{[0,s]}$ and $\tilde{f}_s = 1_{]-\infty,F^{-1}(s)]}$.

To finish the description, a result of Kiefer-Bahadur (See Bahadur (1966)) that says that the addition of the sequences of uniform empirical processes and quantiles processes (6) and (7) is asymptotically, and uniformly on $[0,1]$, zero in probability, that is

$$\sup_{s\in[0,1]} |\alpha_n(s) + \gamma_n(s)| = o_{\mathbb{P}}(1) \text{ as } n \to +\infty. \tag{18}$$

1.3. Our Objective

Based on the notation above, we may say that we search the *GAR* in two steps. First, we want to put the statistics $J_n(s)$ as

$$J_n(s) = \frac{1}{n}\sum_{1\leq j\leq n} h(X_j) + \frac{1}{n}\sum_{1\leq j\leq n}\left\{\mathbb{F}_n(X_j) - F(X_j)\right\} q(X_j) + o_{\mathbb{P}}(n^{-1/2}). \tag{19}$$

Actually, Lo (2013) provided a general **GAR** of the general poverty index *GPI* (See Appendix 2, page 3648). That unified expression *GAR* includes the statistic object we handle here. Here we proceed to a direct investigation in the spirit of Lo et al. (2023). Next, we will have a comparison between our results and those of the *GPI*.

Now, as in Lo et al. (2023), we need the following conditions:
(*CRe1*)[q] $\mathbb{E}q(X) < +\infty$;

(*CRe2*)[q] and, as $n \to +\infty$,

$$\delta_n[q] = \int_0^1 \sqrt{n}\,(s - \mathbb{V}_n(s))\,\Delta_n(q,s)\,ds \to 0$$

with

$$\Delta_n(q,s) = \biggl\{ q\biggl(F^{-1}\left(\mathbb{V}_n(s)\right)\biggr) - q\biggl(F^{-1}(s)\biggr)\biggr\}, \ \ s \in (0,1).$$

Then under *(CRe1)*[q] and *(CRe2)*[q], the *GAR* is turned into

$$J_n = \mathbb{E}h(X) + n^{-1/2}\mathbb{G}_n(h) + n^{-1/2}\int_1^0 \mathbb{G}_n(\tilde{f}_s)\ \ell(s)\ ds + o_{\mathbb{P}}(n^{-1/2}), \tag{20}$$

with

$$\tilde{f}_s = 1_{]-\infty, F^{-1}(s)]},\ s\ (0,1). \tag{21}$$

The general method we will be using has been designed for continuous distribution functions. Here, we require more, i.e.

(HCC): F and F^{-1} are continuous on $[0,+\infty[$ and $[0,\ 1[$ respectively.

Note that this is automatically true if F is continuous and strictly increasing. Here are our results.

2. Our results

Theorem 1. *Suppose that $F(Z) < 1$ and that **(HCC)** holds. Let $s = 1$. Define*

$$D = \biggl\{F(Z)\mathbb{E}(d_Z(X))\biggr\} - \mathbb{E}\biggl(F(X)d_Z(X)\biggr)$$

and, for $t \in \mathbb{R}$,

$$d(t) = \biggl(F(Z)d_Z(t) + \{\mathbb{E}d_Z(X)\}K_Z\biggr) - F(t)d_Z(t).$$

Then, $J_n(1)$ which is the Sen measure has the following GAR

$$J_n(1) = J(1) + n^{-1/2}\mathbb{G}_n(h) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)\ ds + o_{\mathbb{P}}(n^{-1/2}), \tag{22}$$

where

$$\begin{aligned} J(1) &= \frac{2D}{F(Z)}, \\ h(t) &= \frac{2d(t)}{F(Z)} - \frac{2D}{F(Z)^2}K_Z(t), \\ \ell(s) &= \frac{2d_Z\left(F^{-1}(s)\right)}{F(Z)}. \end{aligned}$$

Theorem 2. *Suppose that $F(Z) < 1$ and that **(HCC)** hold. Let $s \geq 1$. We set, for* $(t, s) \in \mathbb{R} \times [0, 1]$,

$$\begin{aligned}
H_1(x) &= (F(Z) - F(x))^s d_Z(x), \quad H_2(x) = s(F(Z) - F(x))^{s-1} d_Z(x) \\
H_3(x) &= F(x)^s K_Z(x), \quad H_4(x) = sF(x)^{s-1} K_Z(x) \\
J(s) &= \frac{\mathbb{E}(K_Z(X))\ \mathbb{E}(H_1(X))}{\mathbb{E}(H_3(X))} \\
h(t) &= \frac{\mathbb{E}(H_1(X))}{\mathbb{E}(H_3(X))} K_Z(t) + \frac{\mathbb{E}(K_Z(X))}{\mathbb{E}H_3(X))} \left(H_1(t) + \{\mathbb{E}H_2(X)\} K_Z(t) \right) \\
&- \frac{\mathbb{E}(K_Z(X))\mathbb{E}(H_1(X))\mathbb{E}(H_3(X))}{\mathbb{E}(H_3(X))^3} \left(H_3(x) + \{\mathbb{E}H_4(X)\} K_Z(x) \right) \\
\ell(s) &= \frac{\mathbb{E}(K_Z(X))}{\mathbb{E}(H_3(X))} H_2\left(F^{-1}(s)\right) - \frac{\mathbb{E}(K_Z(X))\mathbb{E}(H_1(X))\mathbb{E}(H_3(X))}{\mathbb{E}(H_3(X))^3} H_4\left(F^{-1}(s)\right).
\end{aligned}$$

Then, $J_n(s)$ has the following GAR

$$J_n(s) = J(1) + n^{-1/2}\mathbb{G}_n(h) + n^{-1/2} \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)\ ds + o_{\mathbb{P}}(n^{-1/2}), \tag{23}$$

The proofs will show that we also have a second equivalent *GAR* (See the next remark, the notion of equivalent *GAR*s).

Corollary 1. *Suppose that $F(Z) < 1$ and that **(HCC)** hold. Let $s \geq 1$. We set, for* $(t, s) \in \mathbb{R} \times [0, 1]$,

$$\begin{aligned}
H_1(x) &= (F(Z) - F(x))^s d_Z(x), \quad H_2(x) = s(F(Z) - F(x))^{s-1} d_Z(x) \\
H_5(x) &= (F(Z) - F(x))^s K_Z(x), \quad H_6(x) = s(F(Z) - F(x))^{s-1} K_Z(x) \\
J(s) &= \frac{\mathbb{E}(K_Z(X))\ \mathbb{E}(H_1(X))}{\mathbb{E}(H_5(X))} \\
h(t) &= \frac{\mathbb{E}(H_1(X))}{\mathbb{E}(H_5(X))} K_Z(t) + \frac{\mathbb{E}(K_Z(X))}{\mathbb{E}H_5(X))} \left(H_1(t) + \{\mathbb{E}H_2(X)\} K_Z(t) \right) \\
&- \frac{\mathbb{E}(K_Z(X))\mathbb{E}(H_1(X))\mathbb{E}(H_5(X))}{\mathbb{E}(H_5(X))^3} \left(H_5(x) + \{\mathbb{E}H_6(X)\} K_Z(x) \right) \\
\ell(s) &= \frac{\mathbb{E}(K_Z(X))}{\mathbb{E}(H_5(X))} H_2\left(F^{-1}(s)\right) - \frac{\mathbb{E}(K_Z(X))\mathbb{E}(H_1(X))\mathbb{E}(H_5(X))}{\mathbb{E}(H_5(X))^3} H_6\left(F^{-1}(s)\right).
\end{aligned}$$

Then, $J_n(s)$ has the following GAR

$$J_n(s) = J(1) + n^{-1/2}\mathbb{G}_n(h) + n^{-1/2} \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)\ ds + o_{\mathbb{P}}(n^{-1/2}). \tag{24}$$

Remark 1. By doing $s = 1$ in the *GAR* of Theorem 2, we should get equivalent *GAR*s in Theorems 1 and 2 and in Corollary 1. Let us consider the three *GAR*s which are equivalent, i.e., which concern the same index J estimated by J_n:

$$J_n = J_{(1)} + n^{-1/2}\mathbb{G}_n(h_1) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell_1(s)\ ds + o_{\mathbb{P}}(n^{-1/2});$$
$$J_n = J_{(2)} + n^{-1/2}\mathbb{G}_n(h_2) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell_2(s)\ ds + o_{\mathbb{P}}(n^{-1/2});$$
$$J_n = J_{(3)} + n^{-1/2}\mathbb{G}_n(h_3) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell_3(s)\ ds + o_{\mathbb{P}}(n^{-1/2}),$$

where $\ell_i(s) = -q_i(F^s)$, $s \in [0,1]$ and $\mathbb{E}h_i(X)^2$, $\mathbb{E}q_i(X)^2$ $\mathbb{E}h_i(X)^2 q_i(X)^2$ fine. Then, as limits of J_n, we should have $J_{(1)} = J_{(2)} = J_{(3)}J$. However, we have no reason to have that h_i's are equal, neither that the ℓ_i's are. But for sure, the variance of the

$$\mathbb{G}_n(h_i) + \int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell_i(s)$$

$i \in \{1,2,3\}$ should be equal. In our present analysis, we have the two *GAR*'s. Let us focus only on the index. We have

$$J_{(1)} = 2\frac{F(Z)\mathbb{E}\Big(d_Z(X)\Big) - \mathbb{E}\Big(F(X)d_Z(X)\Big)}{F(Z)};$$
$$J_{(2)} = \frac{F(Z)\mathbb{E}\Big((F(Z)-F(X))d_Z(X)\Big)}{\mathbb{E}(F(X)K_Z(X))};$$
$$J_{(3)} = \frac{F(Z)\mathbb{E}\Big((F(Z)-F(X))d_Z(X)\Big)}{\mathbb{E}((F(Z)-F(X))K_Z(X))}.$$

Since all the convergence conditions hold, we should have: For any distribution function F such that $F(0) = 0$, satisfying **HCC**, for any $Z > 0$ such that $F(Z) < 1$,

$$J_{(1)} = 2\frac{\left(F(Z)\int_0^Z \left(\frac{Z-x}{Z}\right)\ dF(x)\right) - \int_0^Z F(x)\left(\frac{Z-x}{Z}\right)\ dF(x)}{F(Z)},$$

$$J_{(2)} = \frac{F(Z)\int_0^Z (F(Z)-F(x))\left(\frac{Z-x}{Z}\right)\ dF(x)}{\int_0^Z F(x)\ dF(x)}$$

and

$$J_{(3)} = \frac{F(Z)\int_0^Z (F(Z)-F(x))\left(\frac{Z-x}{Z}\right)\ dF(x)}{\int_0^Z (F(Z)-F(x))\, dF(x)}.$$

Our computations lead a common value:

$$J(1) = F(Z) - \frac{2}{Z}\int_0^Z x\left(1-\frac{F(x)}{F(Z)}\right)\ dF(x).$$

As an example, for the standard exponential law: $F(x) = (1-e^{-x})1_{(x\geq 0)}$, that common value is

$$J(1) = 2\left(1-\frac{1}{Z}(1-e^{-Z})\right) - \frac{2Z-e^{-2Z}+4e^{-Z}-3}{2Z(1-e^{-Z})}.$$

Remark 2. In comparison with results in Lo (2013) ((See Appendix 2, page 3648)), our *GARS*s are very much simpler. This showed how needed was that projects. In subsequent papers, that simplicity will help in data-driven applications.

3. Proofs

For once, we notice that for a valid *GAR*, with $\alpha \neq 0$,

$$U_n = \alpha + n^{-1/2}\mathbb{G}_n(L) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ell(s)\ ds + o_{\mathbb{P}}(n^{-1/2}),$$

its inverse has the following *GAR*

$$\frac{1}{U_n} = \frac{1}{\alpha} + n^{-1/2}\mathbb{G}_n\left(\frac{-L}{\alpha^2}\right) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\left(\frac{-\ell(s)}{\alpha^2}\right)\ ds + o_{\mathbb{P}}(n^{-1/2}). \tag{25}$$

3.1. Proofs of the GAR of the Sen index: Theorem 1

Here $s=1$. Let us denote

$$d_Z(x) = \left(\frac{Z-x}{Z}\right)1_{(x\leq Z)},\quad x\geq 0.$$

We have

$$J_n(1) = \frac{2}{n(Q_n+1)}\sum_{j=1}^{n}(Q_n-j+1)\ \ d_Z(X_{j,n}).$$

By using the rank statistic, we get

$$\begin{aligned}
J_n(1) &= \frac{2}{Q_n+1}\sum_{j=1}^{n}\left(\{F_n(Z)+1/n\}-F_n(X_{j,n})\right)\ d_Z(X_{j,n})\\
&= \frac{2}{Q_n+1}\left(\left\{\{F_n(Z)+1/n\}\left\{\sum_{j=1}^{n}d_Z(X_j)\right\}-\left\{\sum_{j=1}^{n}F_n(X_{j,n})\ d_Z(X_{j,n})\right\}\right)\\
&= \frac{2n}{Q_n+1}\left(\left\{\{\mathbb{P}_n(K_Z)+1/n\}\left\{\frac{1}{n}\sum_{j=1}^{n}d_Z(X_{j,n})\right\}-\left\{\frac{1}{n}\sum_{j=1}^{n}F_n(X_{j,n})\ d_Z(X_{j,n})\right\}\right)\\
&= J_n(1,1)\left\{J_n(1,2)-J_n(1,3)\right\}.
\end{aligned} \tag{26}$$

We have

(a) **Law of** Q_n+1

$$\frac{Q_n+1}{n}=\frac{1}{n}\sum_{j=1}^{n}1_{(X_j\leq Z)}+(1/n)=\frac{1}{n}\sum_{j=1}^{n}1_{[0,Z]}(X_j)+o_{\mathbb{P}}(n^{-1/2}).$$

For $K_Z=1_{[0,Z]}$, we have

$$\frac{Q_n}{n}=\mathbb{P}_n(K_Z)=\mathbb{E}K_Z(X)+n^{-1/2}\mathbb{G}_n(K_Z)$$

and then

$$\frac{Q_n+1}{n}=\frac{Q_n}{n}+\frac{1}{n}=\mathbb{E}K_Z(X)+n^{-1/2}\mathbb{G}_n(K_Z)+o_{\mathbb{P}}(n^{-1/2}).$$

and then, by (25)

$$J_n(1,1) = \frac{2}{\mathbb{E}K_Z(X)}+n^{-1/2}\mathbb{G}_n\left(-\frac{2K_Z}{\mathbb{E}K_Z(X)^2}\right)+o_{\mathbb{P}}(n^{-1/2}). \tag{27}$$

Next, we have

$$\begin{aligned}
J_n(1,2) &= \left(\frac{Q_n+1}{n}\right)\left(\frac{1}{n}\sum_{j=1}^{n}\ d_Z(X_j)\right)\\
&= \left(\mathbb{E}(K_Z(X))+n^{-1/2}\mathbb{G}_n(K_Z)+o_{\mathbb{P}}(n^{-1/2})\right)\left(\mathbb{E}d_Z(X)+n^{-1/2}\mathbb{G}_n(d_Z)\right).
\end{aligned}$$

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Then

$$J_n(1,2) = \Big\{\mathbb{E}(K_Z(X))\mathbb{E}(d_Z(X))\Big\} + n^{-1/2}\mathbb{G}_n\Big(\{\mathbb{E}K_Z(X)\}\ d_Z + \{\mathbb{E}(d_Z(X)\}\ K_Z\}\Big) + o_{\mathbb{P}}(n^{-1/2}). \tag{28}$$

We continue in

$$J_n(1,3) = \frac{1}{n}\sum_{j=1}^{n} F(X_{j,n})\ d_Z(X_{j,n}) + \frac{1}{n}\sum_{j=1}^{n} (F_n(X_{j,n}) - F(X_{j,n}))\ \ d_Z(X_{j,n}).$$

So, under the conditions (CRe1)[q] and (CRe2)[q] based on $q(x) = d_Z$ (see Page 3629), we have

$$J_n(1,3) = \mathbb{E}F(X)d_Z(X) + n^{-1/2}\mathbb{G}_n(Fd_Z) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ \ell_1(s))\ ds + o_{\mathbb{P}}(n^{-1/2}), \tag{29}$$

$\ell_1(s) = -d_Z(F^{-1}(s))$ and $\tilde{f}_s = 1_{]0,F^{-1}(s)]}$. Now, by combining , (27), (28) and (29), and by denoting them, respectively,

$$C_j + n^{-1/2}\mathbb{G}_n(A_j) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ L_j(s)\ ds + o_{\mathbb{P}}(n^{-1/2})$$

with $j \in \{1,2,3\}$,

$$\begin{array}{lll}
C_1 = \frac{2}{\mathbb{E}K_Z(X)} & A_1 = -\frac{2K_Z}{\mathbb{E}K_Z(X)^2} & L_1 = 0 \\
C_2 = \Big\{\mathbb{E}(K_Z(X))\mathbb{E}(d_Z(X))\Big\} & A_2 = \Big\{\{\mathbb{E}K_Z(X)\}d_Z + \{\mathbb{E}d_Z(X)\}K_Z\Big\} & L_2 = 0 \\
C_3 = \mathbb{E}\Big(F(X)d_Z(X)\Big) & A_3 = Fd_Z & L_3 = -d_Z\left(F^{-1}(s)\right)
\end{array}$$

Let us set

$$D = \Big\{\mathbb{E}(K_Z(X))\mathbb{E}(d_Z(X))\Big\} - \mathbb{E}\Big(F(X)d_Z(X)\Big).\ \ (= C_2 - C_3)$$

and

$$d = \{\{\mathbb{E}K_Z(X)\}d_Z + \{\mathbb{E}d_Z(X)\}K_Z\Big\} - Fd_Z.\ \ (= A_2 - A_3)$$

The final expansion of

$$\begin{aligned} J_n(1) &= J(1,n)\ \{J_n(2) - J_n(1,3)\} \\ &= C + n^{-1/2}\mathbb{G}_n(A) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ L(s)\ ds + o_{\mathbb{P}}(n^{-1/2}) \end{aligned}$$

with, since

$$C = C_1(C_2 - C_3); A = C_1(A_2 - A_3) + (C_2 - C_3)A_1; L = C_1(L_2 - L_3) + (C_2 - C_3)L_1$$

which gives, as $L_j = 0$ for $j \in 1, 2$, that

$$C = C_1(C_2 - C_3); A = (C_2 - C_3)A_1 - C_1(A_2 - A_3); L = -C_1 L_3.$$

The computation leads to

$$\begin{aligned} C &= \frac{2D}{\mathbb{E}K_Z(X)} \\ A &= \frac{2d}{\mathbb{E}K_Z(X)} - \frac{2D}{(\mathbb{E}K_Z(x))^2}K_Z \\ L &= \frac{2d_Z\left(F^{-1}(s)\right)}{(\mathbb{E}K_Z(X))}. \end{aligned}$$

By replacing $\mathbb{E}K_Z(X) = F(Z)$, we can conclude that the *GAR* (22) in the statement of Theorem 1 whenever *(CRe1)[q]* and *(CRe2)[q]* hold for $q = d_Z$. Since q is bounded (by one), then **(CRe1)[q]** holds. As well, *(CRe2)[q]* is checked in Appendix 1, Page 3645. The theorem is completely proved. □

General case $s > 1$. We have

$$
\begin{aligned}
J_n(s) &= \frac{Q_n}{n\sum_{j=1}^{Q_n} j^s} \sum_{j=1}^{n} (Q_n - j + 1)^s d_Z(X_{j,n}) \\
&= \frac{Q_n}{n\sum_{j=1}^{Q_n} j^s} \times \frac{\frac{1}{n^s}}{\frac{1}{n^s}} \sum_{j=1}^{n} (Q_n - j + 1)^s d(X_{j,n}) \\
&= \frac{Q_n}{n} \left(\sum_{j=1}^{Q_n} \left(\frac{j}{n}\right)^s \right)^{-1} \sum_{j=1}^{n} \left(\frac{Q_n}{n} - \frac{j}{n} + \frac{1}{n} \right)^s d_Z(X_{j,n}) \\
&= \left(\frac{Q_n}{n}\right) \left(\frac{1}{\frac{1}{n}\sum_{j=1}^{Q_n} (\frac{j}{n})^s} \right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\frac{Q_n}{n} - \frac{j}{n} + \frac{1}{n} \right)^s d_Z(X_{j,n}) \right) \\
&=: \frac{A_n \times B_n}{C_n},
\end{aligned}
$$

where

$$
A_n = \left(\frac{Q_n}{n}\right), \ C_n = \frac{1}{n} \sum_{j=1}^{Q_n} \left(\frac{j}{n}\right)^s, \ B_n = \frac{1}{n} \sum_{j=1}^{n} \left(\frac{Q_n}{n} - \frac{j}{n} + \frac{1}{n} \right)^s d_Z(X_{j,n}).
$$

We have

$$
\begin{aligned}
A_n &= \frac{Q_n}{n} = F_n(Z) = \mathbb{P}_n(1_{[0,Z]}) && (30) \\
&= \mathbb{E}(K_Z(X)) + n^{-1/2}\mathbb{G}_n(K_Z). && (31)
\end{aligned}
$$

Let us handle B_n. We set

$$
H_1(x) = (F(Z) - F(x))^s d_Z(x), \quad H_2(x) = s(F(Z) - F(x))^{s-1} d_Z(x), \ x \in \mathbb{R}. \tag{32}
$$

We have

$$
\begin{aligned}
B_n &= \frac{1}{n}\sum_{j=1}^{n}\left(F_n(Z) - F_n(X_{j,n}) + \frac{1}{n}\right)^s d_Z(X_{j,n}) \\
&= \frac{1}{n}\sum_{j=1}^{n}\left(F(Z) - F(X_{j,n})\right)^s d_Z(X_{j,n}) \\
&+ \frac{1}{n}\sum_{j=1}^{n}\left\{\left(F_n(Z) - F_n(X_{j,n}) + \frac{1}{n}\right)^s - \left(F(Z) - F(X_{j,n})\right)^s\right\} d_Z(X_{j,n}) \\
&= \mathbb{P}_n(H_1) \\
&+ \left(\frac{1}{n}\sum_{j=1}^{n}\Delta_n(j)(F(Z) - F(X_{j,n})) - \theta_n\Delta_n(j)\right)^{s-1} s d_Z(X_{j,n}) \\
&= \mathbb{P}_n(H_1) \\
&+ \frac{1}{n}\sum_{j=1}^{n}\Delta_n(j)(F(Z) - F(X_{j,n})))^{s-1} s d_Z(X_{j,n})\Bigg) \\
&+ \left(\frac{1}{n}\sum_{j=1}^{n}W_n(j)\Delta_n(j)\ s d_Z(X_{j,n})\right)
\end{aligned}
\tag{33}
$$

with

$$\Delta_n(j) = \{F_n(Z) - F(Z)\} - \{F_n(X_{j,n}) - F(X_{j,n}\} + \frac{1}{n}.$$

and

$$W_n(j) = \left(F(Z) - F(X_{j,n})) - \theta_n\Delta_n(j)\right)^{s-1} - (F(Z) - F(X_{j,n}))^{s-1}.$$

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Then we have

$$\begin{aligned}
B_n &= \mathbb{P}_n(H_1) \\
&+ \left(\frac{1}{n}(F_n(Z) - F(Z)) \sum_{j=1}^{n} (F(Z) - F(X_{j,n}))^{s-1} \; sd_Z(X_{j,n})\right. \\
&- \frac{1}{n}\sum_{j=1}^{n} (F_n(X_{j,n}) - F(X_{j,n})) \; (F(Z) - F(X_{j,n}))^{s-1} \; sd_Z(X_{j,n}) \\
&+ \left. \frac{1}{n^2}\sum_{j=1}^{n} \; (F(Z) - F(X_{j,n}))^{s-1} \; sd_Z(X_{j,n})\right) \\
&+ \left(\frac{1}{n}\sum_{j=1}^{n} W_n(j)\Delta_n(j)\right) \; sd_Z(X_{j,n}) \\
&= \mathbb{P}_n(H_1) + R_n(1) + R_n(2). \qquad (34)
\end{aligned}$$

We have

$$R_n(1) = \frac{1}{\sqrt{n}}\mathbb{G}_n(K_Z)\,\mathbb{P}_n(H_2) - \frac{1}{\sqrt{n}}\tilde{R}e_n(H_2) + R_n(1,1). \tag{35}$$

Now, remarking that d_Z is bounded (actually, it is in $[0,\ 1]$) and $s \geq 1$ is fixed here. So $\mathbb{E}(F(Z) - F(X))^{s-1}$ is finite. Then the last term, $R_n(1,1)$ is a (big) $O_{\mathbb{P}}(1/n)$ and, then, is a (small) $o_{\mathbb{P}}(n^{-1/2})$. We get

$$R_n(1) = \frac{1}{\sqrt{n}}\mathbb{G}_n(K_Z)\,\mathbb{P}_n(H_2) - \frac{1}{\sqrt{n}}\tilde{R}e_n(H_2) + o_{\mathbb{P}}(n^{-1/2}). \tag{36}$$

To handle $R_n(2)$, we must analyze the $W_n(j)$'s. First, we recall the Kolmogorov-Smirnov law that applies to

$$D_n = \sup_{x \in \mathbb{R}} \sqrt{n}|F_n(x) - F(x)| \rightarrow Z,$$

we have, for $z \geq 0$,

$$\mathbb{P}(Z \leq z) = 1 - 2\sum_{h=1}^{\infty}(-1)^{h-1}\exp(-2h^2 z^2).$$

Hence

$$\sup_{j \in [1,\ Q]} |\Delta_n(j)| = O_{\mathbb{P}}(n^{-1/2}). \tag{37}$$

Now, from this and based on elementary considerations, we have that for $s > 1$, for any $\varepsilon > 0$, for $\lambda > 1$, —for n large enough:

$$V_n = \sup_{1 \leq j \leq Q_n} |W_n(j)| \tag{38}$$

$$\leq \sup_{(s,t) \in [-1,\ 2]^2,\ |s-t| \leq \frac{\lambda}{\sqrt{n}}} \left| s^{s-1} - t^{s-1} \right| = T_n(\lambda) \tag{39}$$

holds with probability greater that $1 - \varepsilon$. Since $T_n(\lambda) \rightarrow 0$ as $n \rightarrow +\infty$, we conclude that $V_n \rightarrow_{\mathbb{P}} 0$.

Next, we get

$$\begin{aligned}|R_n(2)| \ \leq \ & \frac{V_n}{\sqrt{n}}\frac{s}{\sqrt{n}}|F_n(Z)-F(Z)|\sum_{j=1}^{n}\ d_Z(X_{j,n})\\ &+\ \frac{V_n}{\sqrt{n}}\frac{s}{\sqrt{n}}\sum_{j=1}^{n}|F_n(X_{j,n})-F(X_{j,n})|\ d_Z(X_{j,n})\\ &+\ \frac{V_n}{n\sqrt{n}}\frac{s}{\sqrt{n}}\sum_{j=1}^{n}\ d_Z(X_{j,n}).\end{aligned}$$

By using (37) in all three lines, we get that each of the quantities in them is bounded by

$$\frac{sO_{\mathbb{P}}(V_n)}{\sqrt{n}}\left(\frac{1}{n}\sum_{j=1}^{n}\ d_Z(X_{j,n})\right).$$

Since d_Z is bounded by one, we have that

$$R_n(2)=V_nO_{\mathbb{P}}(n^{-1/2})=o_{\mathbb{P}}(n^{-1/2}).$$

Now, from (34), (36) and (3.1), we have

$$J_n(s)=\mathbb{P}_n(H_1)+\frac{1}{\sqrt{n}}\mathbb{G}_n(K_Z)\ \mathbb{P}_n(H_2)-\frac{s}{\sqrt{n}}\tilde{R}e_n(H_2)+o_{\mathbb{P}}(n^{-1/2}). \tag{40}$$

We transform the second term of the sum in the right of that equality as follows:

$$\begin{aligned}&\frac{1}{\sqrt{n}}\mathbb{G}_n(K_Z)\ \mathbb{P}_n(H_2)\\ &=n^{-1/2}\mathbb{G}_n(K_Z)\Big(\mathbb{E}H_2(X)+n^{-1/2}\mathbb{G}_n(H_2)\Big)\\ &=n^{-1/2}\mathbb{G}_n(\{\mathbb{E}H_2(X)\}K_Z)+o_{\mathbb{P}}(n^{-1/2}),\end{aligned} \tag{41}$$

So, we get

$$B_n=\mathbb{E}(H_1(X))+n^{-1/2}\mathbb{G}_n(H_1+\{\mathbb{E}H_2(X)\}K_Z)+n^{-1/2}\tilde{R}e_n(-H_2)+o_{\mathbb{P}}(n^{-1/2}) \tag{42}$$

Finally, we need to treat C_n. But C_n may be written as

$$\begin{aligned} C_n &= \frac{1}{n}\sum_{j=1}^{Q_n}\left(\frac{j}{n}\right)^s && (43)\\ &= \frac{1}{n}\sum_{j=1}^{Q_n}\left(\frac{Q_n-j+1}{n}\right)^s && (44)\\ &= \frac{1}{n}\sum_{j=1}^{n}(F_n(Z)-F_n(X_{j,n})+1/n)^s 1_{(X_{j,n}\leq Z)}, && (45)\end{aligned}$$

where the summation of numbers $j \in \{1, \cdots, n\}$ is done from Q to 1. Hence C_n is a transform of exactly B_n which $d_Z(x)$ by $1_{(0\leq x\leq Z)=K_Z(x)}$. Hence for

$$H_5(x) = (F(Z)-F(x))^s\ K_Z(x),\quad H_6(x) = s(F(Z)-F(x))^{s-1},\quad x\in\mathbb{R},$$

we may reconduct the very similar *GAR*:

$$C_n = \mathbb{E}(H_5(X)) + n^{-1/2}\mathbb{G}_n(H_5 + \{\mathbb{E}H_6(X)\}K_Z) + n^{-1/2}\tilde{R}e_n(-H_6) + o_{\mathbb{P}}(n^{-1/2}) \quad (46)$$

As well, we may directly study C_n as

$$\begin{aligned} C_n &= \frac{1}{n}\sum_{j=1}^{Q_n}\left(\frac{j}{n}\right)^s && (47)\\ &= \frac{1}{n}\sum_{j=1}^{n}F_n(X_j)^s 1_{(X_j\leq Z)}, && (48)\end{aligned}$$

By setting

$$H_3(x) = F(x)^s\ K_Z(x),\quad H_4(x) = sF(x)^{s-1}\ \ K_Z(x),\quad x\in\mathbb{R},$$

and by using the same methods, we get

$$C_n = \mathbb{E}(H_3(X)) + n^{-1/2}\mathbb{G}_n(H_3 + \{\mathbb{E}H_4(X)\}K_Z) + n^{-1/2}\tilde{R}e_n(-H_4) + o_{\mathbb{P}}(n^{-1/2}). \quad (49)$$

To be able to transform B_n and C_n into a CAR, we must have that (CRe1)[H_i] and (CRe2)[H_i] hold for $i \in 2, 4$. We will prove that in Appendix 1, page 3645. So, by (25), we get

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$$
\begin{aligned}
1/C_n &= +o_{\mathbb{P}}(n^{-1/2})\\
&+ \frac{1}{\mathbb{E}(H_3(X))} + n^{-1/2}\mathbb{G}_n\left(-\frac{H_3 + \{\mathbb{E}H_4(X)\}K_Z}{\mathbb{E}(H_3(X))^2}\right)\\
&+ n^{-1/2}\tilde{R}e_n\left(\frac{H_4}{\mathbb{E}(H_3(X))^2}\right).
\end{aligned} \tag{50}
$$

To conclude, we say that the product of three *CAR*:

$$
C_j + n^{-1/2}\mathbb{G}_n(A_j) + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ L_j(s)\ ds + o_{\mathbb{P}}(n^{-1/2})
$$

with $j \in \{1,2,3\}$, is the *CAR*

$$
C + n^{-1/2}\mathbb{G}_n(A + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ L(s)\ ds + o_{\mathbb{P}}(n^{-1/2})
$$

with

$$
\begin{aligned}
&C = C_1C_2C_3\\
&A = C_2C_3A_1 + C_1C_3A_2 + C_1C_2A_3\\
&L = C_2C_3\ L_1 + C_1C_3\ L_2 + C_1C_2L_3
\end{aligned}
$$

Since $L_1 = 0$ as we will see it below, we have

$$
\begin{aligned}
&C = C_1C_2C_3\\
&A = C_2C_3A_1 + C_1C_3A_2 + C_1C_2A_3\\
&L = C_1C_3\ L_2 + C_1C_2L_3
\end{aligned}
$$

Here, the *CAR*'s related to $j \in \{1,2,3\}$ represent those of A_n, B_n and $1/C_n$ respectively. So, for $(x,s) \in \mathbb{R} \times [0,1]$,

$$
\begin{array}{lll}
C_1 = \mathbb{E}(K_Z(X)) & A_1(x) = K_Z & L_1(s) = 0(s)\\
C_2 = \mathbb{E}(H_1(X)) & A_2(x) = H_1(x) + \{\mathbb{E}H_2(X)\}K_Z(x) & L_2(s) = H_2\left(F^{-1}(s)\right)\\
C_3 = \frac{1}{\mathbb{E}(H_3(X))} & A_3(x) = -\frac{H_3(x)+\{\mathbb{E}H_4(X)\}K_Z(x)}{\mathbb{E}(H_3(X))^2} & L_3(s) = -\frac{H_4\left(F^{-1}(s)\right)}{\mathbb{E}(H_3(X))^2}
\end{array}
$$

By applying this to the product of the three *CAR*'s (31), (50) and (42), we finally get *CAR*

$$J_n(s) = C + n^{-1/2}\mathbb{G}_n(A + n^{-1/2}\int_0^1 \mathbb{G}_n(\tilde{f}_s)\ L(s)\ ds + o_{\mathbb{P}}(n^{-1/2}),$$

with, for $(x, s) \in \mathbb{R} \times [0, 1]$,

$$\begin{array}{lcl} H_1(x) &=& (F(Z) - F(x))^s d_Z(x), \quad H_2(x) = s(F(Z) - F(x))^{s-1} d_Z(x) \\ H_3(x) &=& (F(Z) - F(x))^s K_Z(x), \quad H_4(x) = s(F(Z) - F(x))^{s-1} K_Z(x) \\ C &=& \frac{\mathbb{E}(K_Z(X))\ \mathbb{E}(H_1(X))}{\mathbb{E}(H_3(X))} \\ A(x) &=& \frac{\mathbb{E}(H_1(X))}{\mathbb{E}(H_3(X))} K_Z(x) + \frac{\mathbb{E}(K_Z(X))}{\mathbb{E}H_3(X))}\Big(H_1(x) + \{\mathbb{E}H_2(X)\} K_Z(x)\Big) \\ &-& \frac{\mathbb{E}(K_Z(X))\mathbb{E}(H_1(X))\mathbb{E}(H_3(X))}{\mathbb{E}(H_3(X))^3}\Big(H_3(x) + \{\mathbb{E}H_4(X)\} K_Z(x)\Big) \\ L(s) &=& \frac{\mathbb{E}(K_Z(X))}{\mathbb{E}(H_3(X))} H_2\left(F^{-1}(s)\right) - \frac{\mathbb{E}(K_Z(X))\mathbb{E}(H_1(X))\mathbb{E}(H_3(X))}{\mathbb{E}(H_3(X))^3} H_4\left(F^{-1}(s)\right). \end{array}$$

4. Conclusion

The computation of the *GAR*s which are obtained here is more simpler and more practical than those in the study of General Poverty Index in Lo (2013) (See Appendix 2, page 3648), and this by far. In future works, we intend to have finer and more explicit results and to have real-data applications in Africa and North-America.

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APPENDIX 1

We have to show that *(CRe1)[q]* and *(CRe2)[q]* for $q \in \{d_Z, H_2, H_4\}$ (see Pages 3636, 3642) and the definition of $H2$ and H_4, see Theorem 2. All these functions are of the form

$$q(x) = a(x)1_{(0 \leq x \leq Z)},$$

with the function $a(\circ)$ is bounded by one and continuous on compact sets. Then *(CRe1)[q]* is bounded by one for $q \in \{d_Z, H_2, H_4\}$. Now, the following remark will be useful in future steps.

(RCC) Let $a(\circ)$ be continuous functions on interval $[0, B]$, $B > 0$. By using the Assumption **(RCC)**, given lines before the statement of the theorems, we easily see that :

(i) $s \mapsto a(F^{-1}(s))$ is uniformly continuous on interval $[0, b]$, i.e., for $b = F(B) < 1$.
(ii) Consequently $\sup_{0 \leq s \leq b} \ |a(F^{-1<}(s))| < +\infty$.

Now, we have

$$q\left(F^{-1}(s)\right) = a\left(F^{-1}(s)\right) 1_{(0 \leq F^{-1}(s) \leq Z)} = a\left(F^{-1}(s)\right) 1_{(0 \leq s \leq F(Z))}$$

and

$$q\left(F^{-1}(\mathbb{V}_n(s))\right) = a\left(F^{-1}(\mathbb{V}_n(s))\right) 1_{(0 \leq s \leq F_n(Z))}$$

Now, for any $\varepsilon > 0$, for n big enough (say for $n \geq n_0$ for some $n_0 > 1$), we have $(F_n(Z) \leq F(Z) + \varepsilon)$ with probability greater that $1 - \varepsilon$. So, for such big n's, s and $\mathbb{V}_n(s)$ are both in $[0, F(Z) + \varepsilon] \equiv [0, b]$, $0 < b < 1$. For $c_n = \sup_{s \in [0,1]} |\mathbb{V}_n(s) - s|$ (and we know that $c_n \to 0$ [both a.s. and in probability] as $n \to 0$), we have

$$\begin{aligned}
\left|q\left(F^{-1}(s)\right) - q\left(F^{-1}(\mathbb{V}_n(s))\right)\right| &= \left|\left(a\left(F^{-1}(s)\right) - a\left(F^{-1}(\mathbb{V}_n(s))\right)\right) 1_{(s \in [a,b])}\right. \\
&- \left.\left(a\left(F^{-1}(s)\right)\right) 1_{(F(Z) \leq s \ F(Z) + \varepsilon)}\right| \\
&\leq \sup_{(s,t) \in [0,b]^2, |s-t| \leq c_n} \left|a\left(F^{-1}(s)\right) - a\left(F^{-1}(t)\right)\right| \\
&+ \left(a\left(F^{-1}(s)\right)\right) 1_{(F(Z) \leq s \ F(Z) + \varepsilon)}
\end{aligned}$$

Now going back to the definition of *(CRe2)*, Page 3629, and denoting $\gamma_n(s) = \sqrt{n}\,(s - \mathbb{V}_n(s))$ which is such that $d_n = \sup_{s \in [0,1]} |\gamma_n(s)|$ is a (big) $O_{\mathbb{P}}(1)$, we get

$$\delta_n[q] \leq c_n d_n + d_n \int_0^1 \biggl(a\left(F^{-1}(s)\right)\biggr) 1_{(F(Z) \leq s \ F(Z)+\varepsilon)} \ ds.$$

So, for $\delta = (1-F(z))/2 > 0$ (by an assumption of the theorem), for any $0 << \delta\varepsilon$. By the remark **(RCC)** at the beginning of that appendix, we can denote the following finite number

$$V = sup_{s \in [F(Z), F(Z)+\delta]} \biggl| a\left(F^{-1}(s)\right) \biggr| \ \textit{finite}.$$

based on this, we may put more restrictions on ε by setting

$$\varepsilon \leq \min\biggl(\delta, \frac{\eta}{2(1+V)}\biggr), \tag{51}$$

for some fixed $\eta > 0$. With these notation, we have for any $n \geq n_0$,

$$\mathbb{P}\biggl(W_{n,\varepsilon}\biggr) =: \mathbb{P}\biggl(\delta_n[q] \leq c_n d_n + d_n \int_0^1 \biggl(a\left(F^{-1}(s)\right)\biggr) 1_{(F(Z) \leq s \ F(Z)+\varepsilon)} \ ds\biggr) \geq 1-\varepsilon.$$

and hence

$$\mathbb{P}\biggl(W_{n,\varepsilon}\biggr) =: \mathbb{P}\biggl(\delta_n[q] \leq c_n d_n + V d_n \varepsilon\biggr) \geq 1-\varepsilon.$$

Next, for any $\eta > 0$,

$$\begin{aligned}
\mathbb{P}\biggl(\delta_n[q] > \eta\biggr) &= \mathbb{P}\biggl((\delta_n > \eta) \cap W_{n,\varepsilon}\biggr) + \mathbb{P}\biggl((\delta_n[q] > \eta) \cap W_{n,\varepsilon}^c\biggr)\\
&= \mathbb{P}\biggl((\delta_n[q] > \eta) \cap W_{n,\varepsilon}\biggr) + \mathbb{P}\biggl(W_{n,\varepsilon}^c\biggr)\\
&= \mathbb{P}\biggl((\delta_n[q] > \eta) \cap W_{n,\varepsilon}\biggr) + \varepsilon
\end{aligned}$$

Let go further, $c_n = o_{\mathbb{P}(1)}$ and $d_n = O_{\mathbb{P}(1)}$, there exists n_1 such that such that for $n \geq n_1$, $\mathbb{P}(c_n d_n + V d_n < \varepsilon(1+V)) \geq 1-\varepsilon$. Denote $W_n(2) = c_n d_n + V d_n < \varepsilon(1+V)$, we get for $n \geq \max(n_0, n_1)$,

$$\begin{aligned}
\mathbb{P}\biggl(\delta_n[q] > \eta\biggr) &\leq \mathbb{P}\biggl((\delta_n[q] > \eta) \cap W_{n,\varepsilon} \cap W_n(2)\biggr) + \mathbb{P}\biggl((\delta_n[q] > \eta) \cap W_{n,\varepsilon} \cap W_n(2)\biggr) + \varepsilon\\
&\leq \mathbb{P}\biggl((\delta_n[q] > \eta) \cap W_{n,\varepsilon} \cap W_n(2)\biggr) + 2\varepsilon.
\end{aligned}$$

By (51), we have on $W_n(2)$: $\delta_n[q] \leq \eta/2$ and hence $(\delta_n[q] > \eta) \cap W_{n,\varepsilon} \cap W_n(2)$ is an empty set. We finally got, for an $n \geq \max(n_0, n_1)$,

$$\mathbb{P}\biggl(\delta_n[q] > \eta\biggr) \leq 2\varepsilon,$$

which implies

$$\limsup_{n\rightarrow+\infty} \mathbb{P}\biggl(\delta_n[q] > \eta\biggr) \leq 2\varepsilon.$$

which in turn leads to

$$\limsup_{n\rightarrow+\infty} \mathbb{P}\biggl(\delta_n[q] > \eta\biggr) = 0,$$

after we let $\varepsilon \downarrow 0$. ■

APPENDIX 2. We recall here the results of in Lo (2013), for recall purposes and further comparsons. Functions related the Sen measure (resp. Kakwani measure) are under scripted by s (resp. by k).

(a) Sen measure.

$$h_s(y) = \Biggl\{2\left[\left(1-\frac{F_{(1)}(y)}{F_{(1)}(Z)}\right)\left(\frac{Z-y}{Z}\right)\right.$$
$$\left.-\left(\frac{F_{(1)}(y)}{F_{(1)}(Z)}\right)\left(\frac{J_s(F_{(1)})}{F_{(1)}(Z)}\right)\right]+K_s(F_{(1)})\Biggr\}\mathbb{I}_{(y\leq Z)}$$
$$q_s(y) = -\frac{2}{F_{(1)}(Z)}\left[\left(\frac{Z-y}{Z}\right)+\frac{J_s(F_{(1)})}{F_{(1)}(Z)}\right]\mathbb{I}_{(y\leq Z)},$$

with

$$J_s(F_{(1)}) = 2\int_0^{F_{(1)}(Z)}\left(1-\frac{s}{F_{(1)}(Z)}\right)\left(\frac{Z-F_{(1)}^{-1}(s)}{Z}\right)ds,$$

and
$$K_s(F_{(1)}) = 2\left(1-\frac{1}{ZF_{(1)}(Z)}\int_0^{F_{(1)}(Z)}F_{(1)}^{-1}(s)ds\right)+\frac{J_s(F_{(1)})}{F_{(1)}(Z)}.\ \diamond$$

(a) Kakwani's measure.

$$\ell_k(s) = q_k(F^{-1}(s))\ \textit{for}\ 0\leq s\leq 1.$$

$$h_k(y) = \Biggl\{(k+1)\left[\left(1-\frac{F_{(1)}(y)}{F_{(1)}(Z)}\right)^k\left(\frac{Z-y}{Z}\right)\right.$$
$$\left.-\frac{J_k(F_{(1)})}{F_{(1)}(Z)}\left(\frac{F_{(1)}(y)}{F_{(1)}(Z)}\right)^k\right]+K_k(F_{(1)})\Biggr\}\mathbb{I}_{(y\leq Z)},$$

and

$$q_k(y) = -\frac{k(k+1)}{F_{(1)}(Z)}\left[\left(1-\frac{F_{(1)}(y)}{F_{(1)}(Z)}\right)^{k-1}\left(\frac{Z-y}{Z}\right)\right.$$
$$\left.+\frac{J_k(F_{(1)})}{F_{(1)}(Z)}\left(\frac{F_{(1)}(y)}{F_{(1)}(Z)}\right)^{k-1}\right]\mathbb{I}_{(y\leq Z)},$$

where

$$J_k(F_{(1)}) = (k+1)\int_0^{F_{(1)}(Z)} \left(1 - \frac{s}{F_{(1)}(Z)}\right)^k \left(\frac{Z - F_{(1)}^{-1}(s)}{Z}\right) ds$$

and

$$K_k(F_{(1)}) = \frac{k(k+1)}{F_{(1)}(Z)} \int_0^{F_{(1)}(Z)} \left(1 - \frac{s}{F_{(1)}(Z)}\right)^{k-1} \left(\frac{Z - F_{(1)}^{-1}(s)}{Z}\right) ds$$
$$+ \frac{J_k(F_{(1)})}{F_{(1)}(Z)}, \quad \Diamond$$

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