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# Explicit Fourth Order Generalized Jarque-Bera Tests And Applications To Recent Laws

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**Abstract** (Short abstract) Here, we extend the asymptotic theory of the Jarque-Bera (JB) Statistic beyond the normality context and the possibility of using reduced moments of order more than three and four. Our approach is based on recent advances on the empirical function processes. (See 3552 for the full abstract.)

**Key words:** asymptotic distribution; asymptotic statistical tests; normality tests; functional empirical processes; tests for arbitrary distributions
**AMS 2010 Mathematics Subject Classification Objects :** Primary 62E20, 62F05. Secondary 62F12, 62G07.

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**Full Abstract in English**. The jarque-Bera chi-square test is one of the most iconic and celebrated tests of normality. It is overwhelmingly used in many disciplines, especially in Econometrics and related software. Typically, a precise combination of the squared deviations of the empirical kurtotis and the empirical skewness from their true counterparts was proved by Jarque and Bera (1987) to be asymptotically approximated to a chi-square law of two degrees of freedom. The then-derived Jarque-Bera test has been devised from that weal limit. As we will check it again, it proved itself very efficient to detect normality even for single digit values of the sample size. Now, the use of the functional empirical process made it possible to have remarkable expansions of statistics including empirical moments. Thus allows to extend the asymptotic theory of the Jarque-Bera (JB) Statistic beyond the normality context and the possibility of using reduced moments of order more than three and four. The study shed light of coefficients of the classical *JB* statistics paving the way to obtaining generalized tests for fitting arbitrary distributions whenever the finiteness of moments allows it.

**Résumé.** (Abstract in French). Le test de Jarque-Bera a été spécifiquement conçu pour tester la normalité des données et repose naturellement sur les paramètres d'asymétrie et d'applatissement. Dans cette discussion, nous verrons que ce test nécessite en réalité la finitude des huit premiers moments. De plus, en utilisant le processus empirique fonctionnel, nous généraliserons ce test pour ajuster n'importe quelle distribution ayant au moins huit premiers moments finis, ce qui donne naissance aux tests Jarque-Bera généralisés (GJBT). Une étude de simulation sur l'efficacité des GJBT est présentée. De nouvelles pistes de recherche sont recommandées.

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## 1. Introduction

Our aim is to find general Jarque Bera $\chi_2^2$ test for distributions with eight moments at least. Our work will be a generalization of a classical Jarque-Bera test explicitly which designed for testing normality.

For a reminder, the classical Jarque Bera test belongs to the class of omnibus tests, i.e those which assess simultaneously whether the skewness and kurtosis are consistent with a Gaussian model. This test proved optimum asymptotic power and good finite sample properties(see Jarque and Bera (1987)).

A detailed description of that test and related indepth analyses can be found in Bowman (1975), D'Agostino (1971) etc. So we begin by describing that classical Jarque Bera test.

### *1.1. Description of the classical Jarque-Bera test*

Let $X_1$, $X_2$, ... be a sequence of independent real random variables defined on the probability space with the same distribution $F$. We suppose that $F$ has a finite eight-th moment

$$m_8 = \int_{\mathbb{R}} |x|^8 \, dF(x) < \infty.$$

Let us consider the empirical and exact kurtosis and skewness parameters of $F$

$$a_n = \frac{1/n \sum_{i=1}^n (X_i - \bar{X})^4}{1/n \sum_{i=1}^n ((X_i - \bar{X})^2)^2}$$

$$b_n = \frac{1/n \sum_{i=1}^n (X_i - \bar{X})^3}{\left[1/n \sum_{i=1}^n (X_i - \bar{X})^2\right]^{3/2}}$$

$$a = \frac{\mathbb{E}(X - \mathbb{E}X)^4}{\sigma^4} \quad and \quad b = \frac{\mathbb{E}(X - \mathbb{E}X)^3}{\sigma^3}$$

with $\sigma^2 = \mathbb{E}(X - \mathbb{E}X)^2$ is the variance of $X$. The statistic of Jarque-Bera for testing the normal distribution for which $a = 3$ and $b = 0$, is the following

$$J_n = \frac{n}{6}\left(\frac{1}{4}(a_n - 3)^2 + b_n^2\right).$$

The test is one of the most powerful tools for testing normality of a sample. Empirical studies show that this test accepts normality and rejects non normality for sizes around $n = 22$ and $n = 25$. It is obvious that this Jarque-Bera test is based on the first fourth moments

$$m_j = \int_{\mathbb{R}} |x|^j \, dF(x), \quad j = 1, ..., 4.$$

Since it is derived from asymptotic normality status of $a_n$ and $b_n$ the moment $m_j$, $j = 5, ..., 8$ are also required. This secret as revealed by Lo *et al.* (2015) showed that the Jarque-Bera test really relies on the first eight moments and this might explain how much it is efficient for detecting normality and rejecting non normality.

The following example has been given by Lo *et al.* (2015),

$$f(x) = \frac{b}{2\Gamma(a)} \, |x|^{a-1} \, e^{-b|x|}, \quad x \in \mathbb{R}$$

which is the probability distribution function (pdf) of the symmetrized distribution $\gamma_s(a, b)$ of the $\gamma(a, b)$ law. For $b = 1$ and $a = (1 + \sqrt{13})/2$, has the same set of first four moments as a normal distribution

Yet, the Jarque-Bera test rejects the normality of that distribution. The reason might be the difference of the four other moments and those of the normal distribution.

The success of the Jarque-Bera test for testing normality is the main motivation of the work of Lo *et al.* (2015) for getting general forms that ca valid for testing an arbitrary distribution $F$ whenever it has eight moments. The cited study went beyond skewness and kurtosis parameters and considered all the statistical test using functions of $a_n$ and $b_n$.

From this, our motivation is to simplify that study by reproving the results in the specific case inn which we directly use the empirical skewness ans kurtosis.

Next, we will have the opportunity to particularize the study for specific distributions and the work. This is expected to be continued in a handbook.

Our methods as in Lo *et al.* (2015) will be based on the asymptotic theory of the functional empirical proce (fep). The methods relying of the (fep) are discussed by Lo *et al.* (2023), that will be instrumental in our work.

## 2. Our theoretical results

Before we begin let us introduce the notation of the eight moments

$$m_0 = 1, \quad m =: m_1 = \mathbb{E}X \quad and \quad m_j = \mathbb{E}X^j, \ 1 \leq 2 \leq 8.$$

and the functions $h_0(x) = 1$, $h_j(x) = x^j$.

We first have asymptotic normality law of the empirical skewness and kurtosis.

**Theorem 1.** *Let us suppose that the eight-th moments exists. Then we have, as* $n \to +\infty$,

$$\begin{aligned}\sqrt{n}\left((a_n - a), \ (b_n - b)\right) &= (\mathbb{G}_n(C) + \mathbb{G}_n(B)) + o_{\mathbb{P}}(1) \\ &\to Z = (Z_1, \ Z_2) \sim \mathcal{N}((0,0), \Sigma)\end{aligned}$$

*where* $\Sigma_{1,1} = \mathbb{V}ar(C(X))$, $\Sigma_{2,2} = \mathbb{V}ar(B(X))$ *and* $\Sigma_{1,2} = \Sigma_{2,1} = Cov(C(X), B(X))$ *and*

$$\begin{aligned}&(m_2 - m_1^2)^4 C \qquad (1)\\ &= (m_2 - m_1^2)^2 h_4 - 4m_1(m_2 - m_1^2)^2 h_3 \\ &+ \left\{6m_1^2(m_2 - m_1^2)^2 - 2(m_2 - m_1^2)(m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4)\right\} h_2 \\ &+ \left\{(m_2 - m_1^2)^2(-4m_3 + 12m_1m_2 - 12m_1^3) + 4m_1(m_2 - m_1^2)(m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4)\right\} h_1\end{aligned}$$

*and*

$$\begin{aligned}&(m_2 - m_1^2)^3 B \qquad (2)\\ &= (m_2 - m_1^2)^{3/2} h_3 + \left\{-3m_1(m_2 - m_1^2)^{3/2} - (3/2)(m_3 - 3m_1m_2 + 2m_1^3)(m_2 - m_1^2)^{1/2}\right\} h_2 \\ &+ \left\{(m_2 - m_1^2)^{3/2}(-3m_2 + 6m_1^2) + 3m_1(m_3 - 3m_1m_2 + 2m_1^3)(m_2 - m_1^2)^{1/2}\right\} h_1.\end{aligned}$$

From the theorem , we can derive he following corollary:

**Corollary 1.** *Under the hypothesis of the theorem if* $\Sigma$ *is invertible, we have the general Jarque-Bera test:*

*(1) If* $X$ *is symmetrical, then* $Z_1$ *and* $Z_2$ *are independent with* $det(\Sigma) = 0$ *and*

$$J_n = n \ \Sigma_{11}^{-1}(a_n - a)^2 + \Sigma_{22}(b_n - b)^2 \to \chi_2^2,$$

*(2) If* $X$ *is not symmetrical, and if* $det(\Sigma) \neq 0$, *i.e.,* $Z_1$ *and* $Z_2$ *are not independent, then*

$$J_n = n \ \left[\Sigma_{22}(a_n - a)^2 + \Sigma_{11}(b_n - b)^2 - 2\Sigma_{12}(a_n - a)(b_n - b)\right] / \det(\Sigma) \to \chi_2^2,$$

Let us give a simple example:

**Classical Jarque-Bera test with a Gaussian law**. We can re-discover the classical Jarque-Bera test for testing normality. Indeed if $X$ follows a norm law $\mathcal{N}(0,1)$ we have

$$C = -6h_2 + h_4 \ \ and \ \ B = -3h_1 + h_3$$

which leads to

$$\sigma_{11} = 24, \ \ \sigma_{22} = 6 \ \ and \ \sigma_{12} = \sigma_{21} = 0.$$

## 3. Proofs

. In this part, we establish all the computations based on the empirical process function method we recalled in Section . Recall that, for $n \geq 1$,

$$a_n = \frac{1/n \sum_{i=1}^{n}(X_i - \bar{X})^4}{1/n \sum_{i=1}^{n}((X_i - \bar{X})^2)^2} = \frac{\mu_{4,n}}{\mu_{2,n}^2} \tag{3}$$

and

$$b_n = \frac{1/n \sum_{i=1}^{n}(X_i - \bar{X})^3}{1/n \sum_{i=1}^{n}((X_i - \bar{X})^2)^{(3/2)}} = \frac{\mu_{3,n}}{\mu_{2,n}^{(3/2)}}$$

First, we have to expand the numerators $\mu_{3,n}$ and $\mu_{3,n}$ and the denumerators $\mu_{2,n}^2$ and $\mu_{2,n}^4$ in (3) and (3). For the numerators, we have the first expansions as

$$\begin{aligned} \mu_{4,n} &= 1/n \sum_{i=1}^{n}(X_i - \bar{X})^4 \\ &= 1/n \sum_{i=1}^{n}\left(X_i^4 - 4\bar{X}_n \ X_i^3 + 6\bar{X}_n^2 \ X_i^2 - 4\bar{X}_n^3 \ X_i + \bar{X}_n^4\right) \\ &= \mathbb{P}_{X,n}(h_4) - 4\bar{X}_n\mathbb{P}_{X,n}(h_3) + 6\bar{X}_n^2\mathbb{P}_{X,n}(h_2) - 4\bar{X}_n^3\mathbb{P}_{X,n}(h_4)_1 + \bar{X}_n^4 \\ &=: A_n(1) + A_n(2) + A_n(3) + A_n(4) + A_n(5) \end{aligned} \tag{4}$$

and,

$$\begin{aligned} \mu_{3,n} &= \mathbb{P}_{X,n}(h_3) - 3\bar{X}_n\mathbb{P}_{X,n}(h_2) + 3\bar{X}_n^2\mathbb{P}_{X,n}(h_1) - \mathbb{P}_{X,n}(h_3) \\ &=: B_n(1) + B_n(2) + B_n(3) + B_n(4). \end{aligned} \tag{5}$$

Before we complete these expansions with expansions of $A_n(j)$, $j = 1, \cdots, 5$ and $B_n(j)$, $j = 1, \cdots, 4$, we need the following quick expansions below, which by the way, concern expansions of denumerators in (3) and (3). We recall that

$$\mathbb{P}_{X,n}(h_i) = \mathbb{E}h_i(X) + n^{-1/2}\mathbb{G}_n(h_i), \ i \in \{1, \cdots, 4\} \tag{6}$$

and in particular

$$\bar{X}_n = m_1 + n^{-1/2}\mathbb{G}_n(h_1), \tag{7}$$

which, by the delta-method, leads to

$$\begin{aligned} \bar{X}_n^2 &= m_1^2 + n^{-1/2}\mathbb{G}_n(2m_1h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right), \\ \bar{X}_n^3 &= m_1^3 + n^{-1/2}\mathbb{G}_n(3m_1^2\ h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right) \end{aligned} \tag{8}$$

and

$$\bar{X}_n^4 = m_1^4 + n^{-1/2}\mathbb{G}_n(4m_1^3h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right). \tag{9}$$

As well, we have

$$\mu_{2,n} = 1/n\sum_{i=1}^{n}(X_i - \bar{X})^2 = \left(1/n\sum_{i=1}^{n}X_i^2\right) - \bar{X}_n^2 = \mathbb{P}_{X,n}(h_2) - \bar{X}_n^2. \tag{10}$$

By combining (6) and the first expansion in (10), we get

$$\mu_{2,n} = (m_2 - m_1^2) + n^{-1/2}\mathbb{G}_n(h_2 - 2m_1h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{11}$$

By the delta-method, we are lead to

$$\mu_{2,n}^2 = (m_2 - m_1^2)^2 + n^{-1/2}\mathbb{G}_n(\left\{2(m_2 - m_1^2)\right\}\{(h_2 - 2m_1h_1\}) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{12}$$

and

$$\begin{aligned} &\mu_{2,n}^{3/2} \\ &= (m_2 - m_1^2)^{3/2} + n^{-1/2}\mathbb{G}_n(\left\{(3/2)(m_2 - m_1^2)^{1/2}\right\}\{h_2 - 2m_1h_1\}) + o_{\mathbb{P}}\left(n^{-1/2}\right). \end{aligned} \tag{13}$$

So we get the expansions of the quotients in and . Now, let us complete the expansions of the numerator. We begin with:

*Expansions of the* $A_n(j)$, $1 \leq j \leq 5$. First, we have

$$A_n(1) = m_4 + n^{-1/2}\mathbb{G}_n(h_4). \tag{14}$$

Next, by combining (7) and (6) (for $i = 3$), we have

$$A_n(2) = -4m_1m_3 + n^{-1/2}\mathbb{G}_n(-4m_1h_3 - 4m_3h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{15}$$

Next, by combining the first formula in (8) and by (6) (for i=2), we have

$$A_n(3) = 6m_1^2m_2 + n^{-1/2}\mathbb{G}_n(6m_1^2h_2 + 12m_1m_2h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{16}$$

and by combining the second formula in (8) and by (6) for $i = 2$

$$A_n(4) = -4m_1^4 + n^{-1/2}\mathbb{G}_n(-16m_1^3) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{17}$$

and finally, $A_n(5)$ is developed as in (9)

$$A_n(5) = m_1^4 + n^{-1/2}\mathbb{G}_n(4m_1^3\ h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right). \tag{18}$$

By putting together the expansions in (14)-(18), we have

$$\mu_{4,n} \tag{19}$$

$$= m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4$$
$$+n^{-1/2}\mathbb{G}_n(h_4 - 4m_1h_3 + 6m_1^2h_2 + \left\{-4m_3 + 12m_1m_2 - 12m_1^3\right\}h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right),$$

which is

$$\mu_{4,n} = \mu_4 + n^{-1/2}\mathbb{G}_n(C0) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{20}$$

with

$$C0 = h_4 - 4m_1h_3 + 6m_1^2h_2 + \left\{-4m_3 + 12m_1m_2 - 12m_1^3\right\}h_1. \tag{21}$$

*Expansions of the* $B_n(j)$, $1 \leq j \leq 4$. Here, we go quicker from the details done in the expansion of the $A_n(\circ)$'s. We have

$$B_n(1) = m_3 + n^{-1/2}\mathbb{G}_n(h_3). \tag{22}$$

Next, by combining (7) and (6) (for $i = 2$), we have '

$$B_n(2) = -3m_1m_2 + n^{-1/2}\mathbb{G}_n(-3m_1h_2 - 3m_2h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{23}$$

Next, by combining the first formula in (8) and by (6) (for i=1), we have

$$B_n(3) = 3m_1^3 + n^{-1/2}\mathbb{G}_n(9m_1^2h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{24}$$

and, finally, $B_n(4)$ is expanded as in the second formula in (8) that is

$$B_n(4) = -m_1^3 + n^{-1/2}\mathbb{G}_n(-3m_1^2h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right). \tag{25}$$

The compilation of expansions (22)-(25) leads to

$$\mu_{3,n} \tag{26}$$

$$= m_3 - 3m_1m_2 + 2m_1^3 + n^{-1/2}\mathbb{G}_n(h_3 - 3m_1h_2 + \left\{-3m_2 + 6m_1^2\right\}h_1) + o_{\mathbb{P}}\left(n^{-1/2}\right),$$

which is

$$\mu_{3,n} = \mu_3 + n^{-1/2}\mathbb{G}_n(B0) + o_{\mathbb{P}}\left(n^{-1/2}\right) \tag{27}$$

with

$$B0 = h_3 - 3m_1h_2 + \left\{-3m_2 + 6m_1^2\right\}h_1. \tag{28}$$

Now, we may proceed to final expansions.

*Expansions of* $a_n$. From (3), we have

$$\mu_{2,n}^2 = \mu^2 + n^{-1/2}\mathbb{G}_n(C1) + o_{\mathbb{P}}\left(n^{-1/2}\right) \ \textit{with}\ C1 = \left\{2(m_2 - m_1^2)\right\}\left\{(h_2 - 2m_1h_1)\right\}.$$

So, by Lemma 2 in Lo *et al.* (2023), we have

$$a_n = \frac{\mu_4 + n^{-1/2}\mathbb{G}_n(C0) + o_{\mathbb{P}}\left(n^{-1/2}\right)}{(m_2-m_1)^2 + n^{-1/2}\mathbb{G}_n(C1) + o_{\mathbb{P}}\left(n^{-1/2}\right)} = a + n^{-1/2}\mathbb{G}_n\left(C\right) + o_{\mathbb{P}}\left(n^{-1/2}\right),$$

with

$$C = \frac{1}{(m_2-m_1)^2}C0 - \frac{m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4}{(m_2-m_1)^4}C1.$$

*Expansions of* $b_n$. From (3), we have

$$\mu_{2,n}^{(3/2)} = \mu^{(3/2)} + n^{-1/2}\mathbb{G}_n(B1) + o_{\mathbb{P}}\left(n^{-1/2}\right) \ \textit{with}\ B1 = \left\{(3/2)(m_2-m_1^2)^{1/2}\right\}\left\{(h_2 - 2m_1h_1)\right\}.$$

Similarly, we get

$$b_n = \frac{m_3 - 3m_1m_2 + 2m_1^2 + n^{-1/2}\mathbb{G}_n(B0) + o_{\mathbb{P}}\left(n^{-1/2}\right)}{(m_2-m_1)^{(3/2)} + n^{-1/2}\mathbb{G}_n(B1) + o_{\mathbb{P}}\left(n^{-1/2}\right)} = b + n^{-1/2}\mathbb{G}_n\left(B\right) + o_{\mathbb{P}}\left(n^{-1/2}\right),$$

with

$$B = \frac{1}{(m_2-m_1)^{(3/2)}}B0 - \frac{m_3 - 3m_1m_2 + 2m_1^2}{(m_2-m_1)^3}B1.$$

Lengthy computations lead to

$$\begin{aligned}
&(m_2-m_1^2)^4C \qquad (29)\\
&= (m_2-m_1^2)^2h_4 - 4m_1(m_2-m_1^2)^2h_3\\
&+ \left\{6m_1^2(m_2-m_1^2)^2 - 2(m_2-m_1^2)(m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4)\right\}h_2\\
&+ \left\{(m_2-m_1^2)^2(-4m_3 + 12m_1m_2 - 12m_1^3) + 4m_1(m_2-m_1^2)(m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4)\right\}h_1
\end{aligned}$$

and

$$\begin{aligned}
&(m_2-m_1^2)^3B \qquad (30)\\
&= (m_2-m_1^2)^{3/2}h_3 + \left\{-3m_1(m_2-m_1^2)^{3/2} - (3/2)(m_3 - 3m_1m_2 + 2m_1^3)(m_2-m_1^2)^{1/2}\right\}h_2\\
&+ \left\{(m_2-m_1^2)^{3/2}(-3m_2 + 6m_1^2) + 3m_1(m_3 - 3m_1m_2 + 2m_1^3)(m_2-m_1^2)^{1/2}\right\}h_1.
\end{aligned}$$

## 4. Implementation of the method and Simulation studies

### *4.1. Implementing the test*

Here, we are interested in a global implementation of the statistical $\chi_2^2$-test we derived in Corollary 1. Let us describe how we can use it in the software R. Given that we want to validate that some data $X = (X_/, \cdots, X_n)$ is a sample of size $n \geq 1$ from a probability law whose *cdf* is $F$, having at least eight first moments $M[j]$, $1 \leq j \leq 8$, we proceed as follows.

(1) Transforming the data if necessary: since the *GJBT* is invariant to scaling and positioning, we may transform the data as $(\circ - B)/A)$, $0 < A \in \mathbb{R}$ and $B \in \mathbb{R}$ and test data $Z_j = (X_j - B)/A$.

(2) Sitting a script in R for charging the vector $M = (M[1], \cdots, M[0])$ in which parameters, if any, should be handled (fixed or estimated). Here after, we give a script with examples of a Gaussian law and a Binomial law:

**Assigning moments**.

```
#Use Momemts of a Gaussian Model
M <- numeric()
M[1]=0
M[2]=1
M[3]=0
M[4]=3
M[5]=0
M[6]=15
M[7]=0
M[8]=105

#Use Momemts of a Bernoulli Model
p=0.3
M <- numeric()
M[1]=p
M[2]=p
M[3]=p
M[4]=p
M[5]=p
M[6]=p
M[7]=p
M[8]=p
```

(2) Computing the function $C$ and $B$ in Corollary 1, denoted here as $C2$ and $B2$, as well as the skewness ($b$) and the kurtosis ($a$) parameters through the following script:

```
## Compiler les moments de la loi a tester
## Rappel: hp(x)=x^p
## Variance and stabdard deviation
(sigmaC=M[2]-(M[1]^2))
(sigma=sqrt(sigmaC))

## muX expressions
muX(2,M)= M[2]-M[1]^2
muX(3,M)= M[3]-3*M[1]*M[2]+2*M[1]^3
muX(4,M)= M[4]-4*M[1]*M[3]+6*M[1]^2*M[2]-3*M[1]^4

## Auxilliary functions
A2 <- function(x) h2(x)-2*M[1]*h1(x)
A3 <- function(x) h3(x)-3*M[1]*h2(x)-(3*M[2]-6*M[1]^2)*h1(x)
A4 <- function(x) h4(x)-4*M[1]*h3(x)+6*M[1]^2*h2(x)
                  +(-4*M[3]+12*M[1]*M[2]-12*M[1]^3 )*h1(x)

## Functions C and B
B2 <- function(x) sigma^(-3)*( A3(x)-(3/2)*sigma^{-2}*muX(3,M)*A2(x))
C2 <- function(x) sigma^(-4)*( A4(x) - 2*sigma^{-2}*muX(4,M)*A2(x))

## skewness and kurtosis paameters
(a=muX(4,M)/(muX(2,M)^2))
(b=muX(3,M)/(muX(2,M)^(3/2)))
```

(3) Computing the variances of $C(X)$ and $B(X)$ and their co-variance. We recommend here the Monte-Carlo method whenever we have a quick generator of data X. This practice can be done independently. Since we need a fitting test for a particular law $F$, we may compute these parameters with the highest sizes $N$ of samples of generated data with also high values of the replication numbers $B$. When these parameters are most correctly approximated, the test is ready to be used either for simulations or statistical tests. However, when available, direct computations and even expressions (as in Gaussian laws or symmetrical laws) are better. Here is the script of the Monte-Carlo method.

```
#VAR(C2(X), VAR(B2(X), covar(C2(X),B2(X))
#Monte-Carlo
B=10000
N=1000
j=1
varC=0
varB=0
covarCB=0
Z<- numeric()

for(j in 1:B){
        Z=rnorm(N, 0,1)
        varC=varC + (var(C2(Z))/B)
        varB=varB + (var(B2(Z))/B)
        covarCB=covarCB+(cov(C2(Z),B2(Z))/B)
}
varC
varB
covarCB
(detCB=(varC*varB)-(covarCB^2))
```

Now, everything is ready easy for the simulation or for the statistical test.

**(4a) Simulations: validating the performance of the tests**. In the simulation test, we fix sizes $n \leq 1$ and we generate $B$ samples of that size from the true distribution $F$ and we report the mean $p$-value denoted as *pval* as in the following script.

```
n=11
pval=0
B=100
for(h in 1:B){
        #generer X: n observations de la loi courante
        (sigmac=var(X))
        (sigma=sqrt(sigmac))
        (moyX=mean(X))
        an=sum((X-moyX)^4)/n
        (an=an/(sigma^4))
         (bn=sum((X-moyX)^3)/n)
        (bn=bn/(sigma^3))
        (jbg=varB*(an-a)^2)
        (jbg=jbg+varC*((bn-b)^2))
        (jbg=jbg-(2*covarCB*(an-a)*(bn-b)))
        (jbgCurrent=n*jbg/detCB)
        pval=pval+((1-pchisq(jbgCurrent,2))/B)
        #pval=pval+(1-pchisq(jbgCurrent,2))
}
pval
```

**(4b) Application of the test**. Once the test is adopted satisfactorily and we want to test that some data $Y$ of size $n$ can be fitted to $F$, we get a one-time computation of the $p$-value as follows.

```
X=... (data to be fitted to F)
n=... (size of data)
(sigmac=var(X))
(sigma=sqrt(sigmac))
(moyX=mean(X))
an=sum((X-moyX)^4)/n
(an=an/(sigma^4))
(bn=sum((X-moyX)^3)/n)
(bn=bn/(sigma^3))
(jbg=varB*(an-a)^2)
(jbg=jbg+varC*((bn-b)^2))
(jbg=jbg-(2*covarCB*(an-a)*(bn-b)))
(jbgCurrent=n*jbg/detCB)
pval=(1-pchisq(jbgCurrent,2))
```

### *4.2. Results of the study*

We have mixed results with two strong trends.

(1) This *GJBT* is extremely powerful in detecting the true distributions for even very low sizes as $n = 10$, even smaller than that. So, we understand why the Gaussian version is used so much in area as medical research where data may be hardly assembled.

(2) This *GJBT* has significant weakness in rejecting reasonable alternatives unless great sizes of $n$ are required. May be the power in detecting has the side effect to also validate near *distributions*.

**A comment**. The jarque-Bera test is used a lot in Ordinary Linear regression requiring the Gaussian data. At the light the two point below, a rejection of the normality really means that the data is very far from that normality.

**A recommendation**. We strongly recommend the reader who wants to apply this tool to proceed a large simulation study using the sizes of his data and to check that his work alternatives are rejected.

## 5. Conclusion, recommendations and perspectives

Without any doubt the *GJBT* is, for us, a major breakthrough in statistical fitting distributions with enough moments. It is general, simple to be implemented even for non-symmetrical laws. It might be more useful when the side-effect of accepting obviously different laws. We still have hope to handle this side-effect since this test use the skewness and the kurtosis only. We will try combinations of high order *GJBT*'s based on Lo *et al.* (2015) in that task.

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