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A wavelet-based estimation of the calibration function in conditional copula model

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Abstract. In the presence of covariates, the dependence structure of a vector of random variables can be modelled by using conditional copula function. If the latter belongs to a parametric copula family, say $\mathfrak{C}_\theta$, an important question is how the dependence parameter θ is related to these covariates. In this paper, we propose a wavelet-based regression approach to estimate the relationship between θ and some real-valued covariate. We consider wavelet shrinkage estimators and show their performance via a simulation study. An application to a meteorological dataset reveals that the temperature influences the dependence structure between the maximum and the minimum relative humidity variables, whenever it takes either very large values or very small values.

Key words: conditional copula; calibration function; dependence parameter; wavelet regression.
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Résumé (Abstract in French) En présence de covariables, la structure de dépendance entre variables aléatoires peut être modélisée à l'aide d'une fonction de copule conditionnelle. Si cette dernière appartient à une famille de copules paramétriques, $\mathfrak{C}_\theta$, une question importante est de savoir comment le paramètre de dépendence de copule θ est lié à ces covariables. Dans cet article, nous proposons une approche de régression par ondelettes pour estimer la relation entre le paramètre de dépendance θ et une certaine covariable réelle observée en même temps que les variables d'intérêt. Nous considérons des estimateurs non linéaires d'ondelettes et montrons leur performance à travers une étude de simulation. Une application à des données météorologiques révèle que la température influence la structure de dépendance entre les variables d'humidité relative maximale et minimale, dès qu'elle prend des valeurs assez grandes ou assez petites.

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1. Introduction

Currently, copulas are widely used for modeling dependence structure between random variables. They have been applied in various domains such as : finance, insurance, survival analysis and meteorology. A large class of parametric copula models, describing different types of dependence, are introduced in the literature. However, when the dependence structure of a given random vector is influenced by the values of another measured covariate, it is convenient to deal with the conditional copula model. In this paper, we are interested in estimating non-parametrically the functional relationship between the dependence parameter, θ, of a parametric conditional copula model, and some real co-variate X. In the literature, this relationship is often described as

$$\theta(X) = g^{-1}(\eta(X)), \qquad \text{i.e.,} \qquad g(\theta(X)) = \eta(X), \tag{1}$$

where g^{-1} is a known inverse link function ensuring that the dependence parameter, θ, considered as a function of the covariate X, takes values in the correct range; η is the so-called *calibration function* which adjusts the level of dependence on the covariate values.

Since the extension of Sklar's theorem Patton (2006) to conditional distribution functions, bringing more flexibility to copulas, dependence modeling via conditional copula function has gained an increasing interest amongst researchers. For instance, dealing with the Clayton copula family, Craiu (2009) proposed a parametric approach, where the dependence parameter θ is a simple linear function in the covariate X . He utilized the maximum likelihood method to estimate the coefficient in the linear relation. Assuming known marginal distributions, Acar *et al.*(2011) provided a nonparametric approach based on local polynomial techniques to estimate the calibration function η within a local likelihood framework. This approach has been extended by Abegaz *et al.*(2012) to the case of unknown marginal distributions.

In the same spirit, Zou (2015) proposed a penalized estimation approach that allows parsimonious and enhanced interpretation of dependence structures of random variables; whereas Sabeti (2013) employed cubic splines in a Bayesian framework. By letting the marginal distribution functions unspecified, Gijbels *et al.*(2011) also proposed a testing methodology with various parametric forms for the calibration function η.

In this paper, we propose a wavelet-based estimation approach to the calibration function η. Indeed, wavelet series allow parsimonious expansion of various types of functions. Because of their good localization properties, wavelet bases adapt well to local features of many kinds of functions, including inhomogeneous and discontinuous ones. The approximation properties of wavelet bases are discussed at length in Härdle *et al.*(1998). For more details on wavelet theory, we refer to Daubechies (1992), Mallat (1989), Meyer (1992), Vidakovic (1999) and references therein.

The methodology of this paper employs a regression model which is based on a binning procedure. Precisely, we deal with a fixed design wavelet regression model, where the response variable is defined by the quantity $Z := \eta(X) = g(\theta(X))$, and the predictor is the covariate X. We first partitionne the support of the covariate X into a finite number m of bins and construct, for each bin, an empirical value representing the function parameter $\theta(\cdot)$ in that bin. Then, since the link function g is known, we will apply it to these empirical values (associated with the bins) to obtain a series of observations of the function η, which we considered here as a theoretical signal corrupted by an additive noise. This allows us to use wavelet shrinkage techniques to estimate the true calibration function η.

The paper is organized as follows. Section 2 describes the methodology. After recalling some facts on the wavelet expansion on the interval [0,1], we present the binning procedure along with Mallat's pyramidal algorithm. We also discuss in this section asymptotic minimax properties of the linear and nonlinear wavelet shrinkage estimators. In Section 3, we make simulation experiments to first study the effect of the binning, with different bin sizes, on the performance of the estimators, and then to compare the different parametric copula families utilized. Section 4 presents an application to a real dataset, while Section 5 concludes the paper.

2. Methodology

2.1. Wavelet expansion on the interval

If the function η belongs to $L^2([0,1])$, the space of measurable and square integrable functions defined on $[0,1]$, we can use wavelet bases on the interval $[0,1]$ to expand η. Let ϕ be a scaling function and ψ be its associated mother wavelet. Assume that both of ϕ and ψ are compactly supported, and for all integers $j \in \mathbb{Z}$, $k \in \mathbb{Z}$, introduce the functions

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k); \qquad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Let $j_0 \in \mathbb{N}$ be a fixed, in Cohen *et al.*(1993) it is constructed an orthonormal wavelet basis for the space $L^2([0,1])$, with exactly 2^j basis functions at each scale $j \geq j_0$. Precisely, the family $\{\phi_{j_0,k} : k = 0, \ldots, 2^{j_0} - 1\} \bigcup \{\psi_{j,k} : j \geq j_0, k = 0, \ldots, 2^j - 1\}$ forms an orthonormal basis for $L^2([0,1])$. Thus, our calibration function η can be decomposed as follows:

$$\eta(x) = \sum_{k=0}^{2^{j_0}-1} \alpha_{j_0,k}\phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x), \quad x \in [0,1], \tag{2}$$

where $\alpha_{j_0,k} = \int_0^1 \phi_{j_0,k}(x)\eta(x)dx$ and $\beta_{j,k} = \int_0^1 \psi_{j,k}(x)\eta(x)dx$ are respectively the scaling and detail coefficients; the parameter j_0 is called a resolution level.

Note that the orthonormal bases proposed in Cohen *et al.*(1993) are boundary adapted. That is, the corresponding wavelet transform automatically handles the boundary effects. There are also other methods for correcting the boundary bias of wavelet estimators such as : periodization, symmetrization and zero padding.

2.2. A binning procedure

Consider n independent and identically distributed observations $\{(Y_{1i}, Y_{2i}, X_i)\}_{i=1}^n$ of a random triple (Y_1, Y_2, X). Suppose that X is a continuous covariate, with support $[a,b], -\infty < a < b < \infty$, and (Y_1, Y_2) is a continuous random couple, with marginal distributions F_1 and F_2, whose dependence structure is influenced by the values of the covariate X. Let H_x, F_{1x} and F_{2x} denote respectively the joint and marginal conditional distributions given $X = x$ of the pair (Y_1, Y_2). The conditional version of Sklar's Theorem says that : for any x in the support of X there exists a unique copula function C_x such that

$$H_x(y_1, y_2) = C_x(F_{1x}(y_1), F_{2x}(y_2)), \quad y_1, y_2 \in \mathbb{R}. \tag{3}$$

C_x represents the dependence structure extracted from the joint conditional distribution H_x. In this paper, we will assume that C_x belongs to a given parametric copula family $\mathfrak{C}_\theta$; that is, its form does not change with the co-variate values x. However, the x values adjust the level (or strength) of dependence , which is measured by the parameter θ, so that we can write $C_x(u_1, u_2) \equiv C(u_1, u_2, \theta(x))$, with $\theta(x)$ satisfying model (1).

If the conditional margins F_{1X} and F_{2X} are known, then we deal with real observations $U_{1i} = F_{1X}(Y_{1i}/X_i)$ and $U_{2i} = F_{2X}(Y_{2i}/X_i)$ drawn from the model

$$(U_{1i}, U_{2i})/X_i \sim C(u_1, u_2; \theta(X_i)), \tag{4}$$

where $\theta(X_i) = g^{-1}(\eta(X_i))$, for $i = 1, \ldots, n$.
If F_{1X} and F_{2X} are unknown, one can replace U_{1i} and U_{2i} by pseudo-observations as, for example, in Abegaz *et al.*(2012) :

$$\hat{U}_{1i} = \hat{F}_{1X}(Y_{1i}|X_i), \qquad \hat{U}_{2i} = \hat{F}_{2X}(Y_{2i}|X_i), \quad \text{for } \ i = 1, \ldots, n, \tag{5}$$

where for $j = 1, 2$ and $\mathbb{I}(\cdot)$ denoting the indicator function,

$$\hat{F}_{jX}(y|x) = \sum_{i=1}^{n} w_{n,i}(x, h)\mathbb{I}(Y_{ji} \leq y),$$

with

$$w_{n,i}(x, h) = \frac{K_h(X_i - x)}{\sum_{k=1}^{n} K_h(X_k - x)}, \qquad \text{and} \qquad K_h(\cdot) = \frac{1}{h}K\left(\frac{\cdot}{h}\right),$$

where h denotes a bandwidth controlling the smoothness and $K(\cdot)$ is a symmetric kernel function. Recall that our aim is to determine the relationship between the copula parameter $\theta(\cdot)$ and the co variate X; that is to estimate the calibration function $\eta(\cdot)$. The definition of the quantity $Z := \eta(X) = g(\theta(X))$ suggests us using a regression framework. But, we do not have direct observations of the random variable Z, which depends upon the functional copula parameter $\theta(\cdot)$. In order to use a regression setting, we will construct, for the random variable Z, a series of observations based on a suitable finite grid of points located in the support $[a, b]$ of X.

Let $\Delta > 0$ be a fixed real number. Let m be a positive integer less than n, and $x_l, l = 1, \ldots, m$ be a finite grid of points defined in such a way that the support $[a, b]$ of X is partitioned into intervals (or bins), I_l, centered at the points x_l with common radius Δ, i.e.

$$I_l \cap I_{l'} = \phi \quad \text{for } l \neq l' \quad \text{and} \quad I_l = \{x \in [a, b] : |x - x_l| \leq \Delta\}, \quad l = 1, \ldots, m.$$

Introduce the blocs of pairwise observations

$$B_l = \{(Y_{1i}, Y_{2i}) : 1 \le i \le n,\ X_i \in I_l\}, \qquad l = 1, \dots, m.$$

To obtain observations for the random quantity $Z = g(\theta(X))$, we have to approximate the unknown parameter function $\theta(\cdot)$ over each bin I_l centered at x_l by a constant. To this end, we first choose Δ small enough so that $\theta(\cdot)$ is invariant within each bin I_l; i.e., $\theta(x) = \theta(x_l) =: \theta_l$, for all $x \in I_l$, where θ_l is a real constant that would be the true copula parameter value if the covariate X were only restricted in the interval I_l. Then, we estimate each θ_l based only on the pairewise observations (Y_{1i}, Y_{2i}) in the corresponding bloc B_l of size n_l. This procedure yields a series of estimators, say $\hat{\theta}_l, l = 1, \dots, m$ approximating the parameter function $\theta(\cdot)$ locally over the different bins $I_l, l = 1, \dots, m$.

To define $\hat{\theta}_l, l = 1, \dots, m$, we use the Kendall's tau inversion method. We first compute the empirical Kendall's tau within the bloc B_l, and then invert the theoretical Kendall's tau formula (pertaining to the considered copula family) to get an estimate $\hat{\theta}_l$ for each θ_l. Thus, applying the link function g, we obtain a series of empirical values, say $Z_l := g(\hat{\theta}_l), l = 1, \dots, m$, that may be considered as independent random observations of the quantity $Z = g(\theta(X))$.
Now for each $l = 1, \dots, m$, it is clear that $\hat{\theta}_l$ is a consistent estimator of θ_l. Indeed, since the variables Y_1 and Y_2 are supposed to be continuous, the empirical Kendall's tau within the bloc B_l, defined as $\hat{\tau}_l = 2/C^2_{n_l} \sum_{i<j} \delta_{ij} - 1$, with $\delta_{ij} = \mathbb{I}(Y_{1i} < Y_{1j}, Y_{2i} < Y_{2j}) + \mathbb{I}(Y_{1i} > Y_{1j}, Y_{2i} > Y_{2j})$, is a consistent estimator of τ_l (the theoretical Kendall's tau of the population from which the bloc B_l is drawn). By inversion of $\hat{\tau}_l$, we get $\hat{\theta}_l$ which is also a consistent estimator of θ_l. Finally, as g is known, recalling the definition of Z_l, a consistent estimator of $g(\theta_l)$, say $\widehat{g(\theta_l)}$, is given by

$$\widehat{g(\theta_l)} = g(\hat{\theta}_l) = Z_l,$$

But within each bin we have $g(\theta_l) = g(\theta(x_l)) = \eta(x_l)$ in view of relation (1). Hence

$$\widehat{\eta(x_l)} = Z_l.$$

This suggests us considering the following regression model :

$$Z_l = \eta(x_l) + \varepsilon_l, \quad l = 1, \dots, m, \tag{6}$$

where $\varepsilon_l, l = 1, \dots, m$, are i.i.d random errors with zero mean and variance σ^2 ; and $\eta(\cdot)$ represents the *calibration function* that we want to recover here nonparametrically.
To this end, we propose a wavelet shrinkage approach. That is, we will consider the function $\eta(\cdot)$ as a theoretical signal, of which, noisy observations are given by a realization of the random series $Z_l = g(\hat{\theta}_l), l = 1, \dots, m$; and we will denoise this series by using wavelet transforms. For fixed design models, it is usually

assumed without loss of generality that the sample points x_l are within the unit interval $[0,1]$ and are equidistant, for example $x_l = \frac{l}{m}, l = 1, \ldots, m$. Furthermore, the number of sample points x_l should be a power of 2, i.e. $m = 2^J,\ J \in \mathbb{N}^*$. These assumptions allow to perform both the Discrete Wavelet Transform (DWT) and its inverse (IDWT) using Mallat (1989) pyramidal algorithm.

Let W be the orthogonal transform matrix associated with a given wavelet basis. Then applying this algorithm yields an approximation $\hat{\eta}$ of the calibration function η after the following steps :

1. Consider a sequence $z = (z_1, \ldots, z_m)$ of realizations of $(Z_1, \ldots, Z_m)$;
2. Apply the forward DWT to obtain a vector of wavelet coefficients : $\omega = Wz$;
3. Apply a thresholding function $\delta(\cdot)$ to obtain the estimated coefficients : $\hat{\omega} = \delta(\omega)$;
4. Apply the inverse IDWT to obtain an approximation of the function η over the grid-points : $\hat{\eta} = W^T\hat{\omega}$, where W^T denotes the transpose of W, and $\hat{\eta} = (\hat{\eta}_1, \ldots, \hat{\eta}_m)$ is a vector of m components approximating the function η over the grid-points, i.e. $\hat{\eta}_l = \widehat{\eta(x_l)}, l = 1, \ldots, m$.

2.3. Minimax properties

In this section we discuss asymptotic minimax properties of wavelet-based estimators of η in both cases of linear and nonlinear shrinkage rules. Without loss of generality the support $[a,b], -\infty < a < b < \infty$ of the covariable X can be reduced to $[0,1]$ by applying the transformation: $(X-a)/(b-a)$.

2.3.1. Linear wavelet estimator

Consider the pairewise observations $(x_l, Z_l), l = 1, \ldots, m = 2^J$, J positive integer, from model (6). Then natural estimators for the scaling coefficients $\alpha_{j_0,k}$ and the detail coefficients $\beta_{j,k}$ can be respectively defined as

$$\widehat{\alpha}_{j_0,k} = \frac{1}{m}\sum_{l=1}^{m} Z_l\phi_{j_0,k}(x_l) \quad \text{and} \quad \widehat{\beta}_{j,k} = \frac{1}{m}\sum_{l=1}^{m} Z_l\psi_{j,k}(x_l), \quad j \geq j_0.$$

Given a resolution level $j_m \geq j_0$, the linear shrinkage rule "kills" all the detail coefficients in decomposition (2) from level j_m by posing :

$$\widehat{\beta}_{j,k} = 0,\ j \geq j_m,\ k = 0, 1, \ldots, 2^j - 1.$$

This, results in the linear wavelet estimator of η:

$$\widehat{\eta}_{j_m}(x) = \sum_{k=0}^{2^{j_0}-1} \widehat{\alpha}_{j_0,k}\phi_{j_0,k}(x) + \sum_{j=j_0}^{j_m}\sum_{k=0}^{2^j-1} \widehat{\beta}_{j,k}\psi_{j,k}(x) = \sum_{k=0}^{2^{j_m}-1} \widehat{\alpha}_{j_m,k}\phi_{j_m,k}(x), \qquad x \in [0,1],$$

which corresponds to the estimation of the orthogonal projection of function η onto the sub-space V_{j_m} element of the multi-resolution analysis $(V_j)_{j\in\mathbb{Z}}$ generated by the father wavelet ϕ.

The optimality, in the minimax sense, of linear wavelet estimators is often investigated over Besov function classes and for L_p-risks, $0 < p \leq \infty$. Under certain regularity conditions including: $2^{j_m} \simeq m^{\frac{1}{2s+1}}$, the linear wavelet estimator $\widehat{\eta}_{j_m}$ attains the optimal rate of convergence, which is of the order $O(m^{\frac{-s}{2s+1}})$, over the Besov balls $B^s_{r,q}(M)$ of radius $M > 0$, with $s > 0$, $0 < p, q \leq \infty$. For more details see, e.g., Donoho *et al.*(1996), Härdle *et al.*(1998).

A major drawback of the linear shrinkage rule is that the optimal rate of convergence depends on the regularity s of the function η, which is unknown in practice. This rule is thus not appropriate, when the function η is not very regular. For such functions one usually relies on nonlinear shrinkage rules.

2.3.2. Nonlinear wavelet estimator

Let j_0, j_m be two resolution levels, both of them depending on m, with $j_m > j_0$. There are two popular ways to define nonlinear (or thresholding) wavelet estimators : hard-thresholding and soft-thresholding rules. Given a threshold $t > 0$, the nonlinear wavelet estimator of η is generally defined as

$$\widehat{\eta}^*_m(x) = \sum_{k=0}^{2^{j_0}-1} \widehat{\alpha}_{j_0,k}\varphi_{j_0,k}(x) + \sum_{j=j_0}^{j_m-1}\sum_{k=0}^{2^j-1} \gamma^*(\widehat{\beta}_{jk}, t)\psi_{j,k}(x), \quad x \in [0,1],$$

where $\gamma^*(\cdot, t)$ is a threshold function defined as

$$\gamma^*(y,t) = \begin{cases} sgn(y)(|y|-t)\mathbb{I}(|y|>t) & \text{for soft-thresholding,} \\ y\mathbb{I}(|y|>t) & \text{for hard-thresholding} \end{cases}$$

where $sgn(y)$ designs the sign of y and $\mathbb{I}(A)$ is indicator of a set A.

The optimality, in the minimax sense, of nonlinear wavelet estimators has also been investigated over Besov function classes and for L_p-risks. The additional hypothesis, compared to the linear case, is that the support of the function to be estimated is compact. It is proved (see, e.g. Donoho *et al.*(1996), Härdle *et al.*(1998)) that nonlinear wavelet estimators are near optimal (up to a logarithmic factor) under certain conditions.

3. Simulation study

In this section we evaluate the finite sample performance of our wavelet approach. To this end, we compute the integrated square error (ISE) of the wavelet shrinkage estimator and report the mean after B replications. We take the covariate X to be uniformly distributed in $[0,1]$, and deal with three parametric copula families: Clayton, Gumbel and Frank, with parameter function $\theta(X) = g^{-1}(\eta(x))$, where g^{-1} is a specific link function and η is the calibration function chosen arbitrarily in these simulations. Note that the link function is $g^{-1}(t) = \exp(t)$ for Clayton copula, $g^{-1}(t) = \exp(t) + 1$ for Gumbel copula and $g^{-1}(t) = t$ for Frank copula. We also suppose that the laws of the margins Y_1 and Y_2 are influenced by the covariate values x according to the following models : $Y_1 \sim \mathcal{N}(\sin(x), \frac{|x|}{2})$ and $Y_2 \sim \mathcal{E}(e^{x/4})$.

We first generate n values $x_i, i = 1, \ldots, n$ for the covariate X, and then generate n pairs of data $(u_{1i}, u_{2i}); i = 1, \ldots, n$ according to the given parametric copula with parameter $\theta(x_i)$. Finally, we compute $(y_{1i}, y_{2i}); i = 1, \ldots, n$ by using the quantile functions associated with the laws of Y_1 and Y_2, respectively. Simulations are done with two specific forms of calibration function inspired from Acar *et al.*(2011) : a linear form $\eta(x) = x + 1$ and a quadratic form $\eta(x) = 2 - 0.3(x - 2/3)^2$.

The wavelet shrinkage estimator is computed using Mallat's algorithm described in subsection 2.2. A universal soft-thresholding is applied : $t = \sigma\sqrt{2\log m}$, where m is the number of grid-points. As usual, the noise σ is estimated by taking the median absolute deviation of the wavelet coefficients at the finest resolution level, and dividing by 0.6745. All computations are done using the R-package "WaveThresh", which employs the least asymmetric Daubechies' wavelet with 10 vanishing moments. Our performance criterion is the integrated square error (ISE) given by

$$ISE = \frac{1}{m}\sum_{l=1}^{m}(\widehat{\eta(x_l)} - \eta(x_l))^2.$$

We report the mean after $B = 1000$ replications. The effect of bin size (number) on the performance of the estimators is shown in Table 1 for Clayton copula and with two calibration function forms for η: linear and quadratic. One can see that the optimal number of bins depends on the sample size n. The optimal bin size is: $m = 16$ for $n = 500$, $m = 32$ for $n = 1000$,$m = 64$ for $n = 2000$ and $m = 128$ for $n = 4000$. We obtain similar results for Gumbel and Frank copulas.

Table 2 shows the results for the linear specification $\eta(x) = x + 1$, whereas Table 2 displays the results for the quadratic specification $\eta(x) = 2 - 0.3(x - 2/3)^2$. These results show that our wavelet regression approach has a good a performance, when both m and n increase and the ratio $\frac{m}{n}$ tends to a constant. We can also observe that the speed of convergence is faster in the quadratic calibration model than in the linear one. Furthermore, one can also remark that the performance is better in the Clayton and Gumbel cases than in the Frank case. This might be due to the

$n = 500$

Number of bins(m)	8	16	32	64
Linear	0.0604	0.0532	0.1055	0.0716
Quadratic	0.1148	0.0083	0.0406	0.0119

$n = 1000$

Number of bins(m)	8	16	32	64	128
Linear	0.1045	0.0450	0.0120	0.0951	0.0845
Quadratic	0.0652	0.0145	0.0071	0.0140	0.0244

$n = 2000$

Number of bins(m)	8	16	32	64	128	256
Linear	0.0459	0.0300	0.0570	0.0271	0.0689	0.0732
Quadratic	0.1571	0.0121	0.0025	0.0013	0.0105	0.0190

$n = 4000$

Number of bins(m)	8	16	32	64	128	256	512
Linear	0.0313	0.0081	0.0209	0.0416	0.0169	0.0534	0.0488
Quadratic	0.082	0.0489	0.0037	0.0058	0.0025	0.0124	0.0055

Table 1. *Effect of binning on the performance of the estimator, with data generated by Clayton copula.*

fact that the Kendall's tau inverse for Frank copula may not be precise because of the approximation of the Debye function.

(m, n)	$(8, 250)$	$(16, 500)$	$(32, 1000)$	$(64, 2000)$	$(128, 4000)$	$(256, 8000)$
Clayton	0.0891	0.0807	0.0737	0.0712	0.0559	0.0544
Gumbel	0.0761	0.0754	0.0661	0.0610	0.0584	0.0573
Frank	0.2511	0.2474	0.1277	0.0933	0.0924	0.0882

Table 2. *Integrated square error of the wavelet shrinkage estimator in the case of linear calibration :* $\eta(x) = x + 1$.

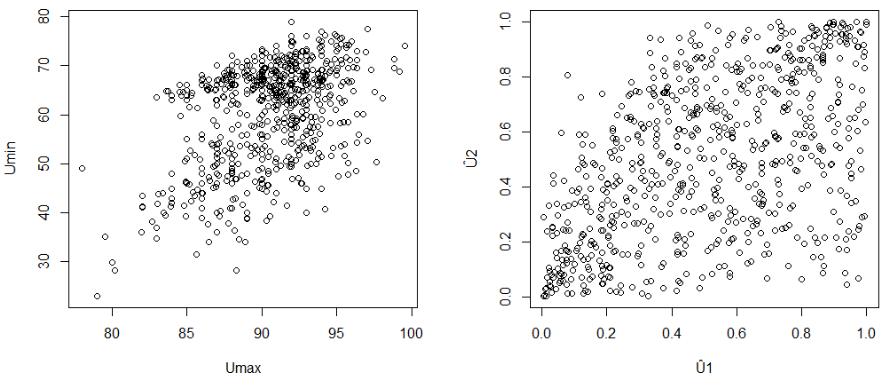
(m, n)	$(8, 250)$	$(16, 500)$	$(32, 1000)$	$(64, 2000)$	$(128, 4000)$	$(256, 8000)$
Clayton	0.0189	0.0107	0.0022	0.0017	0.0015	0.0014
Gumbel	0.0223	0.0207	0.0161	0.0120	0.0100	0.0097
Frank	0.3029	0.1637	0.0805	0.0340	0.0269	0.0121

Table 3. *Integrated square error of the wavelet shrinkage estimator in the case of quadratic calibration :* $\eta(x) = 2 - 0.3(x - 2/3)^2$.

4. Application to real data

In this section we apply our results to meteorological data provided by ANACIM (National Agency for Civil Aviation and Meteorology of Senegal) during the period 1960-2019. The extracted data concern $n = 708$ monthly observations of the following variables : maximum relative humidity ($Umax$), minimum relative humidity ($Umin$) in percentage (%), and maximum temperature ($Tmax$) in Celsius degrees (°C). Our aim is to study the effect of the maximum temperature on the dependence structure between the maximum and the minimum relative humidity variables. That is we want to know if the temperature ($Tmax$) influences the strength of the dependence between the humidity variables $Umax$ and $Umin$?

Figure 1 shows scatter plots of the two humidity variables $Umax$ and $Umin$ and their transformations into uniform scale. It clearly exibits a certain dependence between these two variables. The question is now does the temperature affect this dependence?



(a) (b)

Fig. 1. Scatterplots of (a) the humidity variables $Umax$, $Umin$ and (b) their uniform-scale transforms.

Our covariate $Tmax$ (temperature) takes values in the interval $[20, 40]$, and the sample size $n = 720$. Then, we subdivise the support of $Tmax$ into $m = 2^5 = 32$ bins. We take the binwidth Δ as equal to the range of $Tmax$ divided by the number m of bins . We can now apply the wavelet shrinkage method described in Subsection 2.2 to estimate the calibration function η from which we derive the copula parameter function θ, and then the kendall's tau as a function of the covariate $Tmax$. The results are shown in Figure 2. We also apply the methodology for a number of bins $m = 16$ corresponding to $J = 4$, and obtain similar results.

Figure 2 represents the kendall's tau of the the two humidity variables according to three parametric copulas : Clayton, Gumbel and Frank. It shows that the maximum temperature has actually an effect on the dependence strength between the two humidity variables. This effect is more pronounced whenever the maximum temperature ($Tmax$) takes either very small values or very large values. In contrast, whenever the maximum temperature $Tmax$ takes average values around 28°C, then it weakly influences the strength of dependence between the two humidity variables because the Kendall's tau is minimum.

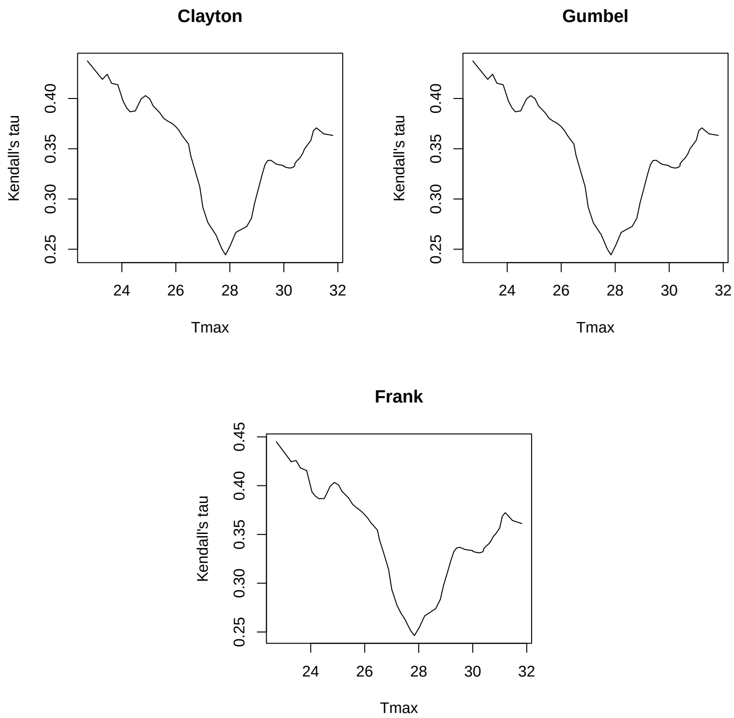


Fig. 2. Wavalet shrinkage estimation of Kendall's tau for Clayton, Gumbel and Frank conditional copulas.

5. Conclusion

In this paper we applied a wavelet regression approach to estimate the relationship between the dependence parameter θ and some real co-variate X in a conditional copula model where the copula family considered is parametric. This approach based on the Mallat's algorithm presents some advantages such as fast computation of the wavelet estimators and their theoretical near optimality over a wide class of regular functions and for a large range of L_p-risks. The results have been applied to a real data-set to analyze the effect of the temperature on the dependence structure between the maximum and minimum relative humidity variables. We found that, the temperature influences the strength of dependence between these two humidity variables, whenever it takes either very larger values or very small values.

However, this temperature-influence is not statistically tested. Thus, performing a general likelihood ratio test to assess the significance of this influence could be an important issue in the future. As well, it might be interesting to compare our approach to that of Acar *et al.*(2011), who utilized a local linear estimation approach for the calibration function η.
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