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On Bayesian estimation in Change-point problems for Poisson Source localization on the plane in non Standard Situation

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Abstract. In this paper, we deal with the problem of estimation of the localization of Poisson signals emitted from a single source of unknown position. More precisely, we are interested by the case of discontinuous (change-point type) model of signal. The asymptotic behavior of the Bayesian estimator of the position of the source in non standard situation related with the uncertainty of the model is studied. In particular the consistency is analyzed. The limit distribution and the convergence of the moments are also described.

Key words: inhomogeneous Poisson process; change-point problem; misspecified model; Bayesian estimator; likelihood ratio process; source localization, sensors.
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Résumé (Abstract in French) Dans cet article, nous traitons du problème de l'estimation de la position d'une source unique émettant des signaux de Poisson à partir d'une source unique de position inconnue. Plus précisément, nous nous intéressons au cas où le modèle du signal est discontinu. On étudie le comportement asymptotique de l'estimateur bayésien de la position de la source en situation non standard liée à l'incertitude du modèle. La consistance est en particulier analysée. La distribution limite et la convergence des moments sont également décrites.

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1. Introduction

This work is devoted to the localization of a source emitting Poisson signals that propagate over an area monitored by many detectors. The detectors receive signals, and based on these detections, the statistician has to estimate the position of the source. Here, we study the properties of Bayesian estimators for the localization of a source. It is more advantageous to use several detectors because the data obtained from a single detector is often incomplete. We refer the interested reader to Magee and Aggarwal(1985) or Chao, Drakopoulos and Lee (1987) for the advantages of using multiple sensors. Due to importance of this problem in many applications, source tracking and localization is a considerable problem that attracted the scientific interest. There exists a large amount of literature for such problems in environmental monitoring, industrial sensing, infrastructure security, military tracking and diverse areas of security and defense. Indeed, the detection of illicit radioactive substances, stored or in transit, has received a great deal of attention by the engineering community. Also the detection of hidden nuclear material by means of sensors is an active area of research as part of defensive strategies. One can cite the work of Baidoo-Williams *and al.*(2015), Liu and Nehorai (2004) and Rao *and al.* (2008). Note that a special case of the source localization problem have been studied in Howse, Ticknor and Muske (2011) where it has described least squares estimation algorithms to estimate the location of a possibly moving source by a fixed number of sensors. In Liu and Nehorai (2004) is presented a technique to locate a source according to Bayesian update methods.

However, the mathematical study of such a problem is not yet sufficiently developped. The present work is the continuity of the study in paper Farinetto, Kutoyants and Top (2020). Our goal is to describe the asymptotic behavior of the Bayesian estimator (BE) of its coordinates through the method developed by Ibragimov and Khasminskii (1981) for the study of such estimators. The same mathematical model can be used in the problem of GPS-localization on the plan (see Luo (2013)). In this case the signals are emitted by k fixed emitters and an object receiving these signals has to define its own position. The algorithms calculating theses positions are based essentially on the adaptive Kalman filtering theory.

The goal of this work is to locate a source by the observations recorded by three detectors. The observations are inhomogeneous Poisson processes with intensity functions depending of the position of the source. Therefore we have three change-point problems where the moments of jumps depend on the distances between the positions of the source and the detectors. The estimation of the position of the source is realized with the help of Bayesian approach. This means that we suppose that the position of the source is a random point with some known distribution. The properties of the corresponding Bayesian estimator are described in the asymptotics of *large intensities*. The formalism used in this work was introduced in the work Farinetto, Kutoyants and Top (2020), where the properties of the Bayesian estimators (change-point case) are described in the case of correctly known model. Also the properties of these estimator constructed by observations from $K \geq 3$ detectors were studied in the regular case (see Chernoyarov and Kutoyants (2020b)) and in the cusp-type singularity case (see Chernoyarov, Dachian and Kutoyants (2020a)). In all these works it was supposed that there is just one source and the beginning of emission is known and the only unknown parameter was the position of the source. The case of unknown beginning of emission and position was studied in Chernoyarov *and al.* (2022a) and the case of two sources in the work Chernoyarov *and al.* (2022b). The statement of the problem of the present work is close to that of Farinetto, Kutoyants and Top (2020) with one essential difference. There, it was supposed that the model is known up to the position of the source and in this work we consider the case, where the position of the source is always unknown but the intensities of the signals used in the construction of the Bayesian estimator are different of the true intensities, i.e., we have the problem of misspecification for Poisson processes models. Similar statements but for different problems were considered in the works of Dabye and Kutoyants (2001) and Dabye, Farinetto and Kutoyants (2003) . Note that those results of Chernoyarov, Dachian and Kutoyants (2020a), Chernoyarov and Kutoyants (2020b), Chernoyarov *and al.* (2022a) and several problems of estimation in the situations of misspecification can be found in book Kutoyants (2023).

2. Statement of the problem

We are interested in locating the source location using a configuration of detectors forming a triangle. Thus, the sequences of measurements collected within the same time window. The measurements from each detector are sent to a central processing unit that combines the data and estimates the coordinates of the source.

The source is located at an unknown position D_0 with coordinates $\vartheta_0 = (x_0, y_0)$ inside a convex set $\Theta \subset \mathcal{R}^2$. Three detectors are placed in the field at known positions at points D_1, D_2, D_3 with the coordinates $\vartheta_j = (x_j, y_j)$, $j \in \{1, 2, 3\}$. Each detector records on a time interval $[0, T]$, $T > 0$, a signal modeled by a Poisson point process $X_j = \{X_j(t),\, 0 \le t \le T\}$, $j \in \{1, 2, 3\}$ of intensity function $\lambda_j(\vartheta_0, t), 0 \le t \le T$. These intensity functions are supposed to be of the form

$$\lambda_j(\vartheta_0, t) = \lambda(t - \tau_j) + \lambda_0, \qquad 0 \le t \le T.$$

Here $\lambda_0 > 0$ is a known intensity of the background noise, $\lambda(t)$ is the unknown intensity function of the signal and $\tau_j = \tau_j(\vartheta_0)$ is the arrival time of the signal to the j-th detector by this formula:

$$\tau_j(\vartheta_0) = \frac{||\vartheta_j - \vartheta_0||}{\nu}, \tag{1}$$

where $||\cdot||$ is the Euclidean norm and ν is the known rate of propagation of the signal in the monitored area. We suppose that $\lambda(t) = 0$ for $t < 0$. At time $t = 0$ the emission of signals begins and τ_j is the arrival time of the signal to the j-th sensor. Since we are interested in the models of observations which allow the estimation with small errors such that $\mathbf{E}_{\vartheta_0}\left(\bar{\vartheta} - \vartheta_0\right)^2 = o(1)$ we use the intensity of the signal taking large values or a periodical Poisson process to describe the data. Then we take the model with large intensity functions $\lambda_j(\vartheta_0, t) = \lambda_{j,n}(\vartheta_0, t)$ which can be written as follows:

$$\lambda_{j,n}(\vartheta_0, t) = n\lambda(t - \tau_j) + n\lambda_0, \qquad 0 \le t \le T. \tag{2}$$

Here n is a *large parameter* and we study estimators as $n \to \infty$.

3. Main results

There are three detectors with coordinates $\vartheta_j = (x_j, y_j)$, $j \in \{1, 2, 3\}$ which measure the particles emitted by some source at the point $\vartheta_0 = (x_0, y_0)$. The observations are modeled by three independent inhomogeneous Poisson processes $X^n = (X_j(t), 0 \le t \le T)$, $j \in \{1, 2, 3\}$ with respective intensity functions

$$\lambda^*_{j,n}(\vartheta_0, t) = n\lambda(t - \tau_j(\vartheta_0))\, \mathbb{I}_{\{t \ge \tau_j(\vartheta_0)\}} + n\lambda_0, \quad 0 \le t \le T, \tag{3}$$

where $\mathbb{I}_A$ stands for the indicator function of a set A, the function $\lambda(\cdot)$ is continuous, strictly positive and $\lambda_0 > 0$. The arrival times of the signals in the j-th sensor according to (1) are $\tau_j = \tau_j(\vartheta_0)$ and the position of the source $\vartheta_0 = (x_0, y_0) \in \Theta \subset \mathbb{R}^2$ will be estimated. We suppose that $\Theta = (\alpha_1, \alpha_2) \times (\beta_1, \beta_2)$ with finite α_i, β_i. The set Θ is bounded, open and convex. Of course, we suppose that for all $\vartheta \in \Theta$ the corresponding $\tau(\vartheta) \in (0, T)$.

Suppose that the signal $\lambda(t), t \geq 0$ is not exactly known but can be bounded below by a constant value $\lambda_1 > 0$ such as

$$0 < \lambda_1 \leq \lambda(t), \qquad 0 \leq t \leq T$$

The Bayesian estimator (BE) is constructed on the basis of the model of observations with the constant intensities of the signal and noise (wrong mathematical model), i.e.,

$$\lambda_{j,n}(\vartheta_0, t) = n\lambda_1 \mathbb{I}_{\{t \geq \tau_j(\vartheta_0)\}} + n\lambda_0, \qquad 0 \leq t \leq T, \tag{4}$$

We study the asymptotic ($n \to \infty$) behavior of the Bayesian estimator of the unknown parameter $\vartheta_0 = (x_0, y_0)$. It is worth noticing that in such misspecified estimation problems, the difference sometimes is not important and the use of a (theoretical) model is justified to esitmate the real position ϑ_0. We refer to Kutoyants (2023), section 5.1.4, where some large class of theoretical model are explained, regarding the change-point problem.

Let us introduce the quantities

$$\underline{\tau} = \min_{j \in \{1,2,3\}} \inf_{\vartheta \in \Theta} \tau_j(\vartheta), \qquad \bar{\tau} = \max_{j \in \{1,2,3\}} \sup_{\vartheta \in \Theta} \tau_j(\vartheta).$$

At this point we have to suppose some conditions ensuring the identifiabilily of the position of the source.

Conditions $\mathcal{R}$:

$\mathcal{R}_1$. *For all* $\vartheta \in \Theta$

$$0 < \underline{\tau} \leq \bar{\tau} < T$$

Consequently we suppose that there exists a small constant $\varepsilon > 0$ *such that for every possible position of the source* $\vartheta_0 \in \Theta$ *and* $j \in \{1, 2, 3\}$

$$\rho_j = ||\vartheta_j - \vartheta_0|| \geq \varepsilon.$$

$\mathcal{R}_2$. *The sensors are not aligned, therefore*

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0.$$

By condition $\mathcal{R}_1$ the case $\tau_j = 0$ is excluded. If the position of the source coincides with the position of one of the sensors, then for this sensor $\tau_j = 0$ and the properties of the estimators will be different. Also suppose that the detectors are on a line on the seashore and the source can be only be located on one side, then two detectors are sufficient for the consistent estimation of the position of the Poisson (radioactive) source.

In fact all the conditions of regularity and identifiability for this model are proposed by Chernoyarov *and al.* (2022a) and Kutoyants (2023), section 5.3, where special attention is payed to condition of identifiability. The pseudo-likelihood $L(\vartheta, X^n)$ is given by (see for example Kutoyants (2023)).

$$\ln L\left(\vartheta, X^{(n)}\right) = \sum_{j=1}^{3} \int_0^T \ln \frac{\lambda_{j,n}(\vartheta,t)}{n\lambda_0} \mathrm{d}X_j(t) - \sum_{j=1}^{3} \int_0^T \left(\lambda_{j,n}(\vartheta,t) - n\lambda_0\right) \mathrm{d}t$$

Recall here, that $\tau_j = \tau_j(\vartheta)$. The Bayesian estimator $\tilde{\vartheta}_n = (\tilde{x}_n, \tilde{y}_n)$ of the parameter $\vartheta_0 = (x_0, y_0)$ with respect to the quadratic loss function is defined by a conditional expectation which can be written as follows

$$\tilde{\vartheta}_n = \mathbf{E}\left(\vartheta / X^{(n)}\right) = \int_\Theta \vartheta p(\vartheta) L\left(\vartheta, X^{(n)}\right) \mathrm{d}\vartheta \left(\int_\Theta p(\vartheta) L\left(\vartheta, X^{(n)}\right) \mathrm{d}\vartheta\right)^{-1}.$$

Note that if the vector ϑ is not random with a given prior density we can use this formula to calculate $\tilde{\vartheta}_n$ which is no more a conditional expectation, but just some way to construct the estimator. In this case it can be called generalized Bayesian estimator Ibragimov and Khasminskii (1981), section I.2 p.23. Therefore, we can take any positive continuous function $p(\theta)$, $\theta \in \Theta$. For example, as the set Θ is bounded, we can put $p(\theta) \equiv 1$.

In order to describe the properties of the Bayesian estimator, we need some additional notations. First let us introduce the unit vectors m_j, for $j \in \{1, 2, 3\}$

$$m_j = \left(\frac{x_j - x_0}{\rho_j}, \frac{y_j - y_0}{\rho_j}\right), \qquad \rho_j = \|\vartheta_j - \vartheta_0\|, \quad \|m_j\| = 1$$

and the sets

$$\mathbb{B}_j = \left\{u \in \mathbb{R}^2 : \quad \langle m_j, u\rangle \geq 0\right\}, \quad \mathbb{B}_j^c = \left\{u \in \mathbb{R}^2 : \quad \langle m_j, u\rangle < 0\right\}.$$

Here $\langle m_j, u\rangle$ denotes the Euclidean scalar product of the vectors m_j and u. The limit likelihood ratio $Z(u), u \in \mathbb{R}^2$ we denote as follows

$$\ln Z(u) = \ell \sum_{j=1}^{3} \left[\Pi_{j,+}(u) \mathbb{I}_{\{u \in \mathbb{B}_j\}} - \Pi_{j,-}(u) \mathbb{I}_{\{u \in \mathbb{B}_j^c\}} \right] - \frac{\lambda_1}{\nu} \langle m_1 + m_2 + m_3, u \rangle,$$

where $\ell = \ln\left(1 + \frac{\lambda_1}{\lambda_0}\right)$, $\Pi_{j,+}(u), u \in \mathbb{B}_j$ and $\Pi_{j,-}(u), u \in \mathbb{B}_j^c$ are independent Poisson random fields such that

$$\mathbf{E}_{\vartheta_0} \Pi_{j,+}(u) = \frac{\lambda_0 \langle m_j, u \rangle}{\nu}, \qquad \mathbf{E}_{\vartheta_0} \Pi_{j,-}(u) = -\frac{(\lambda_0^* + \lambda_0)\langle m_j, u \rangle}{\nu}.$$

Second, we define the random vector $\tilde{\zeta} = (\tilde{\zeta}_1, \tilde{\zeta}_2)$ with the components

$$\tilde{\zeta}_1 = \int_{\mathcal{R}^2} u_1 Z(u)\, \mathrm{d}u \left(\int_{\mathcal{R}^2} Z(u)\, \mathrm{d}u \right)^{-1}$$

and

$$\tilde{\zeta}_2 = \int_{\mathcal{R}^2} u_2 Z(u)\, \mathrm{d}u \left(\int_{\mathcal{R}^2} Z(u)\, \mathrm{d}u \right)^{-1},$$

where $u = (u_1, u_2)$. The main result of this work is the following this theorems where the asymptotic behavior of the estimator $\tilde{\vartheta}_n = (\tilde{x}_n, \tilde{y}_n)$ is described.

Theorem 1. *Let the conditions $\mathcal{R}$ be fulfilled. Then the Bayesian estimator $\tilde{\vartheta}_n$ is uniformly on compact sets $\mathbb{K} \subset \Theta$ consistent: for any $\gamma > 0$*

$$\sup_{\vartheta_0 \in \mathbb{K}} \mathbf{P}_{\vartheta_0} \left(\|\tilde{\vartheta}_n - \vartheta_0\| > \gamma \right) \longrightarrow 0,$$

We have convergence in distribution

$$n\left(\tilde{\vartheta}_n - \vartheta_0\right) \quad \Longrightarrow \quad \tilde{\zeta},$$

and convergence of moments: for any $p > 0$

$$\lim_{n \to \infty} n^p \mathbf{E}_{\vartheta_0} \|\tilde{\vartheta}_n - \vartheta_0\|^p = \mathbf{E}_{\vartheta_0} \| \tilde{\zeta} \|^p,$$

The proof of this theorem is given in the next section. It's based on the general results of Ibragimov and Khasminskii (1981) for the problem of parameter estimation in the case of i.i.d. observations with a discontinuous density function and the application of their results to the study of Bayesian estimators for inhomogeneous Poisson processes (see Kutoyants (1998), Chapter 5). The similar approach was used in the work of Dabye and Kutoyants (2001) for one dimension parameter.

Let us remind the main steps of these proofs. Introduce the normalized likelihood ratio random field

$$Z_n(u) = \frac{L\left(\vartheta_0 + \frac{u}{n}, X^n\right)}{L\left(\vartheta_0, X^n\right)}, \qquad u \in \mathbb{U}_n,$$

where

$$\mathbb{U}_n = \left\{u: \ \vartheta_0 + \frac{u}{n} \in \Theta\right\}.$$

Below we change the variables $\vartheta = \vartheta_0 + \frac{u}{n}$, we have

$$\begin{aligned}\tilde{\vartheta}_n &= \int_\Theta \vartheta \frac{L(\vartheta, X^n)}{L(\vartheta_0, X^n)}\,\mathrm{d}\vartheta \left(\int_\Theta \frac{L(\vartheta, X^n)}{L(\vartheta_0, X^n)}\,\mathrm{d}\vartheta\right)^{-1} \\ &= \vartheta_0 + \frac{1}{n}\int_{\mathbb{U}_n} uZ_n(u)\,\mathrm{d}u \left(\int_{\mathbb{U}_n} Z_n(u)\,\mathrm{d}u\right)^{-1}.\end{aligned}$$

Then

$$n\left(\tilde{\vartheta}_n - \vartheta_0\right) = \int_{\mathbb{U}_n} uZ_n(u)\,\mathrm{d}u \left(\int_{\mathbb{U}_n} Z_n(u)\,\mathrm{d}u\right)^{-1}.$$

Now, if we prove the convergence

$$\begin{aligned}&\left(\int_{\mathbb{U}_n} u_1 Z_n(u)\,\mathrm{d}u, \int_{\mathbb{U}_n} u_2 Z_n(u)\,\mathrm{d}u, \int_{\mathbb{U}_n} Z_n(u)\,\mathrm{d}u\right) \\ &\qquad \Longrightarrow \left(\int_{\mathbb{R}^2} u_1 Z(u)\,\mathrm{d}u, \int_{\mathbb{R}^2} u_2 Z(u)\,\mathrm{d}u, \int_{\mathbb{R}^2} Z(u)\,\mathrm{d}u\right),\end{aligned} \tag{5}$$

we obtain the limit

$$n\left(\tilde{\vartheta}_n - \vartheta_0\right) \Longrightarrow \tilde{\zeta}. \tag{6}$$

To obtain the convergence of moments we have to check the uniform integrability of the random variables $\left\|n\left(\tilde{\vartheta}_n - \vartheta_0\right)\right\|^p$ for any $p > 0$.

4. Proofs

Introduce the normalized likelihood random field

$$\begin{aligned}Z_n(u) \ = \ \exp\Bigg\{&\sum_{j=1}^{3}\int_0^T \ln\frac{\lambda_{j,n}\left(\vartheta_0 + \frac{u}{n}, t\right)}{\lambda_{j,n}\left(\vartheta_0, t\right)}\mathrm{d}X_j(t) \\ &-\sum_{j=1}^{3}\int_0^T \left(\lambda_{j,n}(\vartheta_0 + \frac{u}{n}, t) - \lambda_{j,n}\left(\vartheta_0, t\right)\right)\mathrm{d}t\Bigg\},\end{aligned}$$

where $u \in \mathbb{U}_n$.

Lemma 1. *Let the conditions $\mathcal{R}_1, \mathcal{R}_2$ be satisfied, then the finite dimensional distributions of the process $Z_n(u), u \in \mathbb{U}_n$ converge to the finite dimensional distributions of the process $Z(u), u \in \mathbb{R}^2$ and this convergence is uniform with respect to $\vartheta_0 \in \mathbb{K}$.*

Proof. The characteristic function of $\ln Z_n(u)$ is calculated as follows (see Kutoyants (1998))

$$
\begin{aligned}
\Phi_n(\mu; u) &= \mathbf{E}_{\vartheta_0} \exp\left[i\mu \ln Z_n(u)\right] \\
&= \exp\Bigg\{\sum_{j=1}^{3} \int_0^T \left[\exp\left(i\mu \ln \frac{\lambda_{j,n}\left(\vartheta_0 + \frac{u}{n}, t\right)}{\lambda_{j,n}\left(\vartheta_0, t\right)}\right) - 1\right] \lambda_{j,n}^*\left(\vartheta_0, t\right) \mathrm{d}t \\
&\quad - i\mu \sum_{j=1}^{3} \int_0^T \left(\lambda_{j,n}(\vartheta_0 + \frac{u}{n}, t) - \lambda_{j,n}\left(\vartheta_0, t\right)\right) \mathrm{d}t\Bigg\}.
\end{aligned}
$$

Introduce the sets A_k^n for $k = 1, \cdots, 8$, and $u = (u, v) \in \mathbb{U}_n$

$$
\begin{aligned}
A_1^n &= \{u \in \mathbb{U}_n, \quad \langle u, m_1\rangle \geq 0, \langle u, m_2\rangle \leq 0, \quad \langle u, m_3\rangle \leq 0\}, \\
A_2^n &= \{u \in \mathbb{U}_n, \quad \langle u, m_1\rangle \geq 0, \langle u, m_2\rangle \geq 0, \quad \langle u, m_3\rangle \leq 0\}, \\
A_3^n &= \{u \in \mathbb{U}_n, \quad \langle u, m_1\rangle \geq 0, \langle u, m_2\rangle \geq 0, \quad \langle u, m_3\rangle \geq 0\}, \\
A_4^n &= \{u \in \mathbb{U}_n, \quad \langle u, m_1\rangle \leq 0, \langle u, m_2\rangle \geq 0, \quad \langle u, m_3\rangle \geq 0\}, \\
A_5^n &= \{u \in \mathbb{U}_n, \quad \langle u, m_1\rangle \leq 0, \langle u, m_2\rangle \leq 0, \quad \langle u, m_3\rangle \geq 0\}, \\
A_6^n &= \{u \in \mathbb{U}_n, \quad \langle u, m_1\rangle \leq 0, \langle u, m_2\rangle \leq 0, \quad \langle u, m_3\rangle \leq 0\}. \\
A_7^n &= \{u \in \mathbb{U}_n, \quad \langle u, m_1\rangle \geq 0, \langle u, m_2\rangle < 0, \quad \langle u, m_3\rangle \geq 0\}. \\
A_8^n &= \{u \in \mathbb{U}_n, \quad \langle u, m_1\rangle < 0, \langle u, m_2\rangle \geq 0, \quad \langle u, m_3\rangle < 0\}.
\end{aligned}
$$

Define $\vartheta_u = \vartheta_0 + \frac{u}{n}$, $\tau_j = \tau_j\left(\vartheta_0\right)$, $\rho_j = \nu\tau_j$ and

$$
\tau_j(\vartheta_u) = \frac{1}{\nu}\sqrt{\left(x_j - x_0 - \frac{u}{n}\right)^2 + \left(y_j - y_0 - \frac{v}{n}\right)^2}.
$$

It follows from condition $\mathcal{R}_1$ that $\tau_j(\vartheta_u)$ is differentiable w.r.t. u on $\mathbb{U}_n$. Using the Taylor expansion we obtain

$$
\begin{aligned}
\tau_j(\vartheta_u) &= \tau_j - \frac{u(x_j - x_0) + v(y_j - y_0)}{\nu n \rho_j} + \varepsilon_n(u) \\
&= \tau_j - \frac{1}{\nu n}\langle u, m_j\rangle + \varepsilon_n(u),
\end{aligned}
$$

where $n\varepsilon_n(u) \to 0$ uniformly on compacts u as $n \to \infty$. Thus

$$\tau_j(\vartheta_u) - \tau_j = -\frac{1}{\nu n}\langle u, m_j\rangle + \varepsilon_n(u).$$

Therefore, for all $j = 1, 2, 3$, bounded sets of u and n sufficiently large we have

$$\begin{cases} \tau_j \geq \tau_j(\vartheta_u), & if \quad \langle u, m_j\rangle \geq 0, \\ \tau_j \leq \tau_j(\vartheta_u), & if \quad \langle u, m_j\rangle \leq 0. \end{cases}$$

We will use this fact to calculate the characteristic function $\Phi_n(\mu; u)$ for each set A_k^n , $k = 1, \cdots, 8$ and obtain it limit.

- If $u \in A_1^n$, then $\tau_1 \geq \tau_1(\vartheta_u)$, $\tau_2 \leq \tau_2(\vartheta_u)$ and $\tau_3 \leq \tau_3(\vartheta_u)$. Therefore we can write

$$\begin{aligned}\Phi_n(\mu; u) = \exp\Bigg\{ & \int_{\tau_1(\vartheta_u)}^{\tau_1(\vartheta_0)} \left[\exp\left(i\mu \ln \frac{\lambda_1 + \lambda_0}{\lambda_0}\right) - 1\right] n\lambda_0 \mathrm{d}t - i\mu \int_{\tau_1(\vartheta_u)}^{\tau_1(\vartheta_0)} n\lambda_1 \mathrm{d}t \\ & + \int_{\tau_2(\vartheta_0)}^{\tau_2(\vartheta_u)} \left[\exp\left(i\mu \ln \frac{\lambda_0}{\lambda_0 + \lambda_1}\right) - 1\right] (n\lambda(t - \tau_2(\vartheta_0)) + n\lambda_0) \mathrm{d}t - i\mu \int_{\tau_2(\vartheta_0)}^{\tau_2(\vartheta_u)} -n\lambda_1 \mathrm{d}t + \\ & + \int_{\tau_3(\vartheta_0)}^{\tau_3(\vartheta_u)} \left[\exp\left(i\mu \ln \frac{\lambda_0}{\lambda_0 + \lambda_1}\right) - 1\right] (n\lambda(t - \tau_3(\vartheta_0)) + n\lambda_0) \mathrm{d}t - i\mu \int_{\tau_3(\vartheta_0)}^{\tau_3(\vartheta_u)} -n\lambda_1 \mathrm{d}t \Bigg\}\end{aligned}$$

For $t \in [\tau_j(\vartheta_0), \tau_j(\vartheta_u)]$, $j = 2, 3$ as $n \to \infty$ we obtain

$$\begin{aligned}\lambda(t - \tau_j(\vartheta_u)) &= \lambda(0) + (t - \tau_j(\vartheta_u))\lambda'(0) + o(1) \\ &= \lambda(0) + o(1) \equiv \lambda_0^* + o(1)\end{aligned}$$

Using once again Taylor's expansions by the powers of $\frac{u}{n}$ we obtain the representation

$$\begin{aligned}\lim_{n\to\infty} \Phi_n(\mu; u) = \exp\Bigg\{ & \left[\exp\left(i\mu l\right) - 1\right] \frac{\lambda_0}{\nu} \langle u, m_1\rangle \\ & - \left[\exp\left(-i\mu l\right) - 1\right] \frac{\lambda_0^* + \lambda_0}{\nu} \langle u, m_2 + m_3\rangle \\ & - i\mu \frac{\lambda_1}{\nu} \langle u, m_1 + m_2 + m_3\rangle \Bigg\}.\end{aligned}$$

Comparison of this expression with the form of the characteristic function $\Phi(\mu; u)$ of $\ln Z(u)$ for $u \in A_1^n$ namely

$$
\begin{aligned}
\Phi(\mu;u) &= \mathbf{E}_{\vartheta_0}\exp\left[i\mu\ln Z(u)\right] \\
&= \mathbf{E}_{\vartheta_0}\exp\left[i\mu l\Pi_{1^+}(u)\right]\times\mathbf{E}_{\vartheta_0}\exp\left[i\mu l\Pi_{2^-}(u)\right]\times\mathbf{E}_{\vartheta_0}\exp\left[i\mu l\Pi_{3^-}(u)\right] \\
&\quad\times\exp\left\{-i\mu\frac{\lambda_1}{\nu}\langle u,m_1+m_2+m_3\rangle\right\} \\
&= \exp\Bigg\{\left[\exp(i\mu l)-1\right]\frac{\lambda_0}{\nu}\langle u,m_1\rangle-\left[\exp(-i\mu l)-1\right]\frac{\lambda_0^*+\lambda_0}{\nu}\langle u,m_2+m_3\rangle \\
&\quad-i\mu\frac{\lambda_1}{\nu}\langle u,m_1+m_2+m_3\rangle\Bigg\}
\end{aligned}
$$

shows that we proved for $u\in A_1^n$ the convergence

$$
\mathbf{E}_{\vartheta_0}\exp\left[i\mu\ln Z_n(u)\right]\to\mathbf{E}_{\vartheta_0}\exp\left[i\mu\ln Z(u)\right]
$$

- If $u\in A_2^n$, then similar arguments allow us to verify that

$$
\begin{aligned}
\lim_{n\to\infty}\Phi_n(\mu;u) &= \exp\Bigg\{\left[\exp(i\mu l)-1\right]\frac{\lambda_0}{\nu}\langle u,m_1+m_2\rangle-\left[\exp(-i\mu l)-1\right]\frac{\lambda_0^*+\lambda_0}{\nu}\langle u,m_3\rangle \\
&\quad-i\mu\frac{\lambda_1}{\nu}\langle u,m_1+m_2+m_3\rangle\Bigg\} \\
&= \Phi(\mu;u).
\end{aligned}
$$

- For $u\in A_3^n$ we have

$$
\begin{aligned}
\lim_{n\to\infty}\Phi_n(\mu;u) &= \exp\left\{\left[\exp\left(i\mu\ell\right)-1\right]\frac{\lambda_0}{\nu}\langle u,m_1+m_2+m_3\rangle-i\mu\frac{\lambda_1}{\nu}\langle u,m_1+m_2+m_3\rangle\right\} \\
&= \Phi(\mu;u).
\end{aligned}
$$

For other sets A_k^n we have the corresponding limits. For all sets these limits provide the convergence of characteristic functions

$$
\mathbf{E}_{\vartheta_0}\exp\left[i\mu\ln Z_n\left(u\right)\right]\quad\longrightarrow\quad\mathbf{E}_{\vartheta_0}\exp\left[i\mu\ln Z\left(u\right)\right].
$$

Therefore we have the convergence of one-dimensional distributions. Using the same arguments and helping Kutoyants (2023), Lemme 5.13 it is possible to verify the convergence of the finite-dimensional distributions too, i.e., for any $u_1,\ldots,u_L$ and reals $\mu_1,\ldots,\mu_L$ we have

$$
\mathbf{E}_{\vartheta_0}\exp\left[i\sum_{l=1}^{L}\mu_l\ln Z_n\left(u_l\right)\right]\quad\longrightarrow\quad\mathbf{E}_{\vartheta_0}\exp\left[i\sum_{l=1}^{L}\mu_l\ln Z\left(u_l\right)\right].
$$

Moreover from the presented proofs it follows that the convergence of finite-dimensional distributions is uniform on the compacts $\mathbb{K} \subset \Theta$. In particular,

$$\lim_{n\to\infty} \sup_{\vartheta_0\in\mathbb{K}} \left| \mathbf{E}_{\vartheta_0} \exp\left[i \sum_{l=1}^{L} \mu_l \ln Z_n(u_l) \right] - \mathbf{E}_{\vartheta_0} \exp\left[i \sum_{l=1}^{L} \mu_l \ln Z(u_l) \right] \right| = 0.$$

Further we need the following result.

Lemma 2. *Let the condition $\mathcal{R}_1$ be fulfilled, then for any $R > 0$ and for $u, v \in \mathbb{U}_n$ such that $\|u\| + \|v\| \le R[$, we have*

$$\sup_{\vartheta_0\in\mathbb{K}} \mathbf{E}_{\vartheta_0} \left| Z_n^{\frac{1}{2}}(u) - Z_n^{\frac{1}{2}}(v) \right|^2 \le C \|u - v\|, \tag{7}$$

where $C > 0$.

Proof. Suppose that $u, v \in \mathbb{U}_n$, we have

$$\mathbf{E}_{\vartheta_0} \left| Z_n^{\frac{1}{2}}(u) - Z_n^{\frac{1}{2}}(v) \right|^2 = \mathbf{E}_{\vartheta_0}[Z_n(u)] + \mathbf{E}_{\vartheta_0}[Z_n(v)] - 2\mathbf{E}_{\vartheta_0}\left[Z_n(u)^{1/2} Z_n(v)^{1/2} \right]$$

We will in what follows only consider the case $\tau_j(\vartheta_0) < \tau_j(\vartheta_u) \wedge \tau_j(\vartheta_v)$, where $\wedge$ represents the minimum. The other cases can be treated in a similar way. For the terms on the right-hand side here, the first expectation is

$$\begin{aligned} \mathbf{E}_{\vartheta_0}[Z_n(u)] &= \exp\left\{ \sum_{j=1}^{3} \int^{T} \left(\frac{\lambda_{j,n}(\vartheta_u, t)}{\lambda_{j,n}(\vartheta_0, t)} - 1 \right) \lambda_{j,n}^*(\vartheta_0, t)\mathrm{d}t - \sum_{j=1}^{3} \int_0^T [\lambda_{j,n}(\vartheta_u, t) - \lambda_{j,n}(\vartheta_0, t)]\,\mathrm{d}t \right\} \\ &= \exp\left\{ -n \sum_{j=1}^{3} \int_{\tau_j(\vartheta_0)}^{\tau_j(\vartheta_u)} f_j(t)\mathrm{d}t \right\}, \end{aligned}$$

where

$$f_j(t) = \lambda_1 \left(\frac{\lambda_0 + \lambda\big(t - \tau_j(\vartheta_0)\big)}{\lambda_0 + \lambda_1} - 1 \right). \tag{8}$$

under our situation $f_j(t) > 0$ for $t > 0$. The same way we obtain

$$\mathbf{E}_{\vartheta_0}[Z_n(v)] = \exp\left\{ -n \sum_{j=1}^{3} \int_{\tau_j(\vartheta_0)}^{\tau_j(\vartheta_v)} f_j(t)\mathrm{d}t \right\}, \tag{9}$$

where $f_j(t)$ is defined by (8). For the last expectation, we have

$$\mathbf{E}_{\vartheta_0}\left[Z_n(u)^{1/2}Z_n(v)^{1/2}\right]=\exp\left\{-n\sum_{j=1}^{3}\int_{\tau_j(\vartheta_0)}^{\tau_j(\vartheta_u)\wedge\tau_j(\vartheta_v)}f_j(t)\mathrm{d}t-n\sum_{j=1}^{3}\int_{\tau_j(\vartheta_u)\wedge\tau_j(\vartheta_v)}^{\tau_j(\vartheta_u)\vee\tau_j(\vartheta_v)}g_j(t)\mathrm{d}t\right\},$$

where

$$g_j(t)=\left(1-\left[\frac{\lambda_0}{\lambda_0+\lambda_1}\right]^{\frac{1}{2}}\right)\left(\lambda_0+\lambda\big(t-\tau_j(\vartheta_0)\big)\right)-\frac{\lambda_1}{2}. \tag{10}$$

Simple calculus shows that we have that $g_j(t)>0$ for $t>0$. We can now write

$$\mathbf{E}_{\vartheta_0}\left|Z_n^{\frac{1}{2}}(u)-Z_n^{\frac{1}{2}}(v)\right|^2=\exp\left\{-n\sum_{j=1}^{3}\int_{\tau_j(\vartheta_0)}^{\tau_j(\vartheta_u)\wedge\tau_j(\vartheta_v)}f_j(t)\mathrm{d}t\right\}\times$$
$$\left(\exp\left\{-n\sum_{j=1}^{3}\int_{\tau_j(\vartheta_u)\wedge\tau_j(\vartheta_v)}^{\tau_j(\vartheta_u)}f_j(t)\mathrm{d}t\right\}-2\exp\left\{-n\sum_{j=1}^{3}\int_{\tau_j(\vartheta_u)\wedge\tau_j(\vartheta_v)}^{\tau_j(\vartheta_u)\vee\tau_j(\vartheta_v)}g_j(t)\mathrm{d}t\right\}\right.$$
$$\left.+\exp\left\{-n\sum_{j=1}^{3}\int_{\tau_j(\vartheta_v)\wedge\tau_j(\vartheta_v)}^{\tau_j(\vartheta_u)}f_j(t)\mathrm{d}t\right\}\right)$$

Since $f_j(t)>0$, $g_j(t)>0$, and for all $x>0$, , we have

$$\mathbf{E}_{\vartheta_0}\left|Z_n^{\frac{1}{2}}(u)-Z_n^{\frac{1}{2}}(v)\right|^2\leq 2\left(1-\exp\left\{-n\sum_{j=1}^{3}\int_{\tau_j(\vartheta_u)\wedge\tau_j(\vartheta_v)}^{\tau_j(\vartheta_u)\vee\tau_j(\vartheta_v)}g_j(t)\mathrm{d}t\right\}\right)$$
$$\leq 2n\sum_{j=1}^{3}\int_{\tau_j(\vartheta_u)\wedge\tau_j(\vartheta_v)}^{\tau_j(\vartheta_u)\vee\tau_j(\vartheta_v)}g_j(t)\mathrm{d}t.$$

Therefore, as $g(\cdot)$ is bounded and is positive, we have

$$\mathbf{E}_{\vartheta_0}\left|Z_n^{\frac{1}{2}}(u)-Z_n^{\frac{1}{2}}(v)\right|^2\leq nC_0\sum_{j=1}^{3}\left|\tau_j\left(\vartheta_u\right)-\tau_j\left(\vartheta_v\right)\right|$$

and for large n

$$\left|\tau_j\left(\vartheta_u\right)-\tau_j\left(\vartheta_v\right)\right|\leq\frac{2}{\nu n}\left|\langle u-v,m_j\rangle\right|\leq\frac{C_1}{n}\left\|u-v\right\|.$$

This estimate allows us to write

$$\mathbf{E}_{\vartheta_0}\left|Z_n^{\frac{1}{2}}(u)-Z_n^{\frac{1}{2}}(v)\right|^2\leq C\left\|u-v\right\|.$$

We need also the following lemma

Lemma 3. *Let conditions $\mathcal{R}$ be fulfilled, then for $u \in \mathbb{U}_n$*

$$\sup_{\vartheta_0 \in \mathbb{K}} \mathbf{E}_{\vartheta_0} Z_n^{\frac{1}{2}}(u) \le e^{-\kappa ||u||}, \tag{11}$$

where $\kappa > 0$.

Proof. A direct calculation shows that

$$\mathbf{E}_{\vartheta_0} Z_n^{\frac{1}{2}}(u) = \exp\left\{-n \sum_{j=1}^{3} \int_{\tau_j(\vartheta_0)\wedge\tau_j(\vartheta_u)}^{\tau_j(\vartheta_0)\vee\tau_j(\vartheta_u)} g_j(t)\mathrm{d}t\right\} \le \exp\left\{-nC_2 \sum_{j=1}^{3} |\tau_j(\vartheta_u) - \tau_j(\vartheta_0)|\right\}$$

with $g_j(t)$ in (10) and like

$$n\Big|\tau_j(\vartheta_u) - \tau_j(\vartheta_{\mathbf{0}})\Big| = n\Big|\tau_j(\vartheta_u) - \tau_j\left(\vartheta_0\right)\Big| \ge \Big|\langle m_j, u\rangle\Big| = \left|\left\langle m_j, \frac{u}{\|u\|}\right\rangle\right| \|u\|$$
$$\ge \inf_{\|e\|=1} |\langle m_j, e\rangle| \, ||u|| \ge \kappa_1 \|u\| \,,$$

where $\kappa_1 > 0$. Thus we obtain,

$$\sup_{\vartheta_0 \in \mathbb{K}} \mathbf{E}_{\vartheta_0} Z_n^{\frac{1}{2}}(u) \le e^{-\kappa ||u||}, \tag{12}$$

where $\kappa > 0$. Let $L > 0$, $Q_L = [-L, L] \times [-L, L]$ and define the random variable

$$I_n(L) = \int_{Q_L} u Z_n\left(u\right)\,\mathrm{d}u \left(\int_{Q_L} Z_n\left(u\right)\,\mathrm{d}u\right)^{-1}$$

For arbitrary real numbers α_1, α_2 and α_3, we consider the random variable

$$\Psi_L(Z_n) \equiv \alpha_1 \int_{Q_L} u_1 Z_n(u) du + \alpha_2 \int_{Q_L} u_2 Z_n(u) du + \alpha_3 \int_{Q_L} Z_n(u) du \tag{13}$$

The converge of the integrals (13) is done helping Theorem A.22 in Ibragimov and Khasminskii (1981). Then by the same technique developed by Ibragimov and Khasminskii in the proof of Theorem I.10.2 in Ibragimov and Khasminskii (1981) or Theorem 6.2 in Kutoyants (2023) the lemmas 1, 2 and 3 are sufficient to check whether the assumptions of Theorem A.22 in Ibragimov and Khasminskii (1981). Therefore, the distribution of this random converges to the distribution of the random variable

$$\Psi_L(Z) \equiv \alpha_1 \int_{Q_L} u Z(u) p\left(\vartheta_0\right) du + \alpha_2 \int_{Q_L} v Z(u) p\left(\vartheta_0\right) du + \alpha_3 \int_{Q_L} Z(u) p\left(\vartheta_0\right) du,$$

which implies that for any $L < \infty$, the convergence in distribution

$$I_n(L) \Longrightarrow I(L) \equiv \int_{Q_L} uZ\left(u\right)\,\mathrm{d}u \left(\int_{Q_L} Z\left(u\right)\,\mathrm{d}u\right)^{-1} \tag{14}$$

Now, we need to estimate quantities of the type

$$\int_{Q_L^c} u_1 Z_n(u) du.$$

Therefore we need to study the tail of the process Z_n.

4.1. Estimation of the tails of Z_n

From the definition of stochastic integrals, we have the following representation:

$$\ln Z_n(u) = \sum_{j=1}^{3}\sum_{i_j=1}^{N_j} \ln \frac{\lambda_{j,n}(\vartheta_u, t_{i_j})}{\lambda_{j,n}(\vartheta_0, t_{i_j})} - \sum_{j=1}^{3}\int_0^T \Big[\lambda_{j,n}(\vartheta_u, t) - \lambda_{j,n}(\vartheta_0, t)\Big] dt,$$

where N_j is the number of t_{i_j} of X_j in $[0, T]$. Therefore, we have the jumps of the processes $Z_n^{\frac{1}{4}}(u)$ at the points $u^i \equiv (u_1^i, u_2^i)$ which have along a curve

$$t_{i_j} = \tau_j(\vartheta_{u^i}).$$

Now, we can develop the similar technique as Kutoyants (1998) section 5.2 p. 213, and lemma 3.2 Ibragimov and Khasminskii (1981). We start to take notations: for any integers l and m,

$$\delta_m = [m, m+1] \qquad \delta_l = [l, l+1] \qquad \delta_{m,l} = \delta_m \times \delta_l.$$

set $(u_1, u_2) \in \delta_{m,l}$. We introduce for some $h > 0$ the events:

- $\mathbf{B}_i = \mathbf{B}_i(u_1, u_1 + h)$: the process $Z_n(\cdot, u_2)$, has at least i discontinuities over the rectangle $[u_1, u_1 + h] \times [l, l + 1]$, with $i \in \{1, 2\}$.
- $\mathbf{B}$: the process $Z_n(\cdot, u_2)$ has over the square $\delta_{m,l}$ at least 2 discontinuities and the distance between them is less than $2h$.

We then have the following lemma.

Lemma 4. *There exists positive constants C_1, C_2, C_3 such that*

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1\right\} \leq C_1 h, \qquad \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_2\right\} \leq C_2 h^2$$

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}\right\} \leq C_3 h \tag{15}$$

Proof. A trajectory of the process $Z_n(\cdot, u_2)$ possesses a discontinuity over the rectangle $[u_1, u_1+h] \times [l, l+1]$ only if at least one X_j has a discontinuity on the interval

$$\left]\tau_j\left(x_0+\frac{u_1}{n}, y_0+\frac{u_2}{n}\right), \tau_j\left(x_0+\frac{u_1+h}{n}, y_0+\frac{u_2}{n}\right)\right[\equiv]\tau_j^0, \tau_j^h[, \tag{16}$$

where we suppose $\tau_j^0 \leq \tau_j^h$. Denote by $\mathbf{B}_1^{(j)}$ a event X_j has a discontinuity on the interval (16). Then, we have

$$\mathbf{B}_1 \subset \bigcup_{j=1}^{3} \mathbf{B}_1^{(j)}$$

and using inequality $1-e^{-x} \leq x$, we obtain

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1\right\} \leq \sum_{j=1}^{3} \int_{\tau_j^0}^{\tau_j^h} \lambda_{j,n}^*\left(\vartheta_0, t\right) dt \leq nC\left|\tau_j^h-\tau_j^0\right|$$

Then

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1\right\} \leq C_1 h \tag{17}$$

To estimated $\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_2\right\}$, note that

$$\mathbf{B}_2 \subset \bigcup_{j=1}^{3} \mathbf{B}_2^{(j)} \cup\left(\mathbf{B}_1^{(1)} \cap \mathbf{B}_1^{(2)}\right) \cup\left(\mathbf{B}_1^{(1)} \cap \mathbf{B}_1^{(3)}\right) \cup\left(\mathbf{B}_1^{(2)} \cap \mathbf{B}_1^{(3)}\right)$$

and helping inequality $1-e^{-x}-xe^{-x} \leq x^2$, $x>0$, we obtain

$$\begin{aligned}
\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_2^{(j)}\right\} &= \mathrm{P}_{\vartheta_0}^{(n)}\left\{X_j\left(]\tau_j^0, \tau_j^h[\right) \geq 2\right\} \\
&= 1-\mathrm{P}_{\vartheta_0}^{(n)}\left\{X_j\left(]\tau_j^0, \tau_j^h[\right)=0\right\}-\mathrm{P}_{\vartheta_0}^{(n)}\left\{X_j\left(]\tau_j^0, \tau_j^h[\right)=1\right\} \\
&\leq \left[-\int_{\tau_j^0}^{\tau_j^h} \lambda_{j,n}^*\left(\vartheta_0, t\right) dt\right]^2 \\
&\leq \left(C_1 h\right)^2
\end{aligned}$$

Therefore by independance of X_j we have

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_2\right\} \leq \sum_{j=1}^{3} \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_2^{(j)}\right\} + \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1^{(1)}\right\} \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1^{(2)}\right\} + \\ + \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1^{(1)}\right\} \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1^{(3)}\right\} + \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1^{(2)}\right\} \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_1^{(3)}\right\}$$

and

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_2\right\} \leq C_2 h^2 \tag{18}$$

Subdivise the interval $[m, m+1]$ to $M = [1/h]$ ($[\cdot]$ means interger part) intervals $d_i = (w_i, w_{i+1})$ of length M^{-1}, then any intervall of length h either contained in one of the intervals d_i or belongs to two neighboring intervals d_i, d_{i+1}. Hence

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}\right\} \leq \sum_{i=1}^{M} \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_2(d_i)\right\} + \sum_{i=1}^{M} \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}_2(d_i \cup d_{i+1})\right\},$$

and by (18)

we obtain

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}\right\} \leq C_3 h$$

which completes the proof of Lemma.

Now introduce the notations, which we can to find in Kutoyants (1998), p.214

$$\delta_l(h) = \left\{(v, v', v'') \in \mathbb{R}^3 : l \leq v - h \leq v' \leq v \leq v'' \leq v + h \leq l + 1\right\}$$

and we define $\Delta_h^{m,l}$ by

$$\begin{aligned}
\Delta_h^{m,l}(z) = & \sup_{u,u',u'' \in \delta_m(h)} \left[\min\{\sup_{v\in\delta_l} |z(u,v) - z(u',v)|, \sup_{v\in\delta_l} |z(u,v) - z(u'',v)|\}\right] \\
& + \sup_{v,v',v'' \in \delta_l(h)} \left[\min\{\sup_{u\in\delta_m} |z(u,v) - z(u,v')|, \sup_{u\in\delta_m} |z(u,v) - z(u,v'')|\}\right] \\
& + \sup_{m \leq u \leq m+h} \sup_{v\in\delta_l} |z(u,v) - z(m,v)| \\
& + \sup_{m+1-h \leq u \leq m+1} \sup_{v\in\delta_l} |z(u,v) - z(m+1,v)| \\
& + \sup_{l \leq v \leq l+h} \sup_{u\in\delta_m} |z(u,v) - z(u,l)| \\
& + \sup_{l+1-h \leq v \leq l+1} \sup_{u\in\delta_m} |z(u,v) - z(u,l+1)| \\
= & \; Y_h^m + Y_h^l + Y_0^m + Y_1^m + Y_0^l + Y_1^l,
\end{aligned}$$

with evident notations. Now we begin by estimate the probability $\mathrm{P}_{\vartheta_0}\left\{\Delta_h^{m,l}\left(Z_n^{\frac{1}{4}}(u,v)\right)\geq h^{1/4}\right\}$ where we consider the process Z_n over the square $\delta_{m,l}$. For that, firstly we are going to estimate $\mathrm{P}_{\vartheta_0}\left\{Y_h^m\geq h^{1/4}\right\}$ by follow:

If $\omega\in\mathbf{B}^c$ then the process $Z_n(\cdot,v)$ has one discontinuity at u_{i_0} and then if we put $u_{i_0}=u_1$, the process $Z_n^{1/4}(\cdot,v)$ is continuously differentiable on $]u_1;u_1+h[$ and since $]u_1;u_1+h[$ subset in one d_i and we have for this d_i:

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{u\in d_i}\sup_{u<u''<u+h}\sup_{v\in\delta_l}\left|Z_n^{1/4}(u,v)-Z_n^{1/4}(u'',v)\right|\geq h^{1/4},\mathbf{B}^c\right\}$$

and

$$\begin{aligned}Z_n^{1/4}(u',v)-Z_n^{1/4}(u_1,v)&=\int_{u_1}^{u'}\frac{\partial Z_n^{1/4}(s,v)}{\partial s}ds\\&=\frac{1}{4n}\sum_{j=1}^{3}\int_{u_1}^{u'}Z_n^{1/4}(s,v)\left[\int_0^T\dot{l}(s,t)\left(dX_j(t)-\lambda_j(\vartheta_s,t)dt\right)\right]ds\end{aligned}$$

where we put $l(s,t)=\ln\dfrac{\lambda_{j,n}(\vartheta_s,t)}{\lambda_{j,n}(\vartheta_0,t)}$ and $\vartheta_s=\left(x_0+\dfrac{s}{n};y_0+\dfrac{v}{n}\right)$. Furthermore,

$$\begin{aligned}\sum_{j=1}^{3}\left|\int_0^T\dot{l}(\vartheta_s,t)\left(dX_j(t)-\lambda_j(\vartheta_s,t)dt\right)\right|&\leq\sum_{j=1}^{3}\left|\sum_{i_j=1}^{N_j}\dot{l}(\vartheta_s,t_{i_j})\right|+c_2\\&\leq c_1\sum_{j=1}^{3}X_j(T)+c_2\\&\leq c_1N+c_2.\end{aligned}$$

where the random variable $N=\sum_{j=1}^{3}X_j(T)$. Therefore

$$\sup_{u_1<u'<u_1+h}\sup_{v\in\delta_l}\left|Z_n^{1/4}(u',v)-Z_n^{1/4}(u_1,v)\right|\leq\left(c_1N+c_2\right)\int_{u_1}^{u_1+h}Z_n^{1/4}(s,v)ds\tag{19}$$

Moreover, we have

$$\begin{aligned}
&\mathrm{P}_{\vartheta_0}\left\{Y_h^m \geq h^{1/4}\right\}\\
&= \mathrm{P}_{\vartheta_0}^{(n)}\left\{Y_h^m \geq h^{1/4}, \mathbf{B}\right\} + \mathrm{P}_{\vartheta_0}^{(n)}\left\{Y_h^m \geq h^{1/4}, \mathbf{B}^c\right\}\\
&\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}\right\} + \sum_{i=1}^{M}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{u\in d_i}\sup_{u<u''<u+h}\sup_{v\in\delta_l}\left|Z_n^{1/4}(u,v) - Z_n^{1/4}(u'',v)\right| \geq h^{1/4}, \mathbf{B}^c\right\}\\
&\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\mathbf{B}\right\} + \sum_{i=1}^{M}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\left(c_1N + c_2\right)\sup_{v\in\delta_l}\int_{u_1}^{u_1+h} Z_n^{1/4}(s,v)ds \geq h^{1/4}, \mathbf{B}^c\right\}
\end{aligned}$$

Let us estimate the second probability

$$\begin{aligned}
&\mathrm{P}_{\vartheta_0}^{(n)}\left\{\left(c_1N + c_2\right)\sup_{v\in\delta_l}\int_{u_1}^{u_1+h} Z_n^{1/4}(s,v)ds \geq h^{1/4}, \mathbf{B}^c\right\}\\
&\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\left(c_1N + c_2\right) \geq h^{-1/4}, \mathbf{B}^c\right\}\\
&+\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{v\in\delta_l}\int_{u_1}^{u_1+h} Z_n^{1/4}(s,v)ds \geq h^{1/2}, \mathbf{B}^c\right\}.
\end{aligned}$$

For the first probability we have the following estimate helping Markov inequality

$$\begin{aligned}
\mathrm{P}_{\vartheta_0}^{(n)}\left\{\left(c_1N + c_2\right) \geq h^{-1/4}, \mathbf{B}^c\right\} &\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\left(c_1N + c_2\right) \geq h^{-1/4}\right\}\\
&\leq \mathbf{E}_{\vartheta_0}\left(c_1N + c_2\right)^2 h^{1/2}\\
&\leq Ch^{1/2}
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{v\in\delta_l}\int_{u_1}^{u_1+h} Z_n^{1/4}(s,v)ds \geq h^{1/2}, \mathbf{B}^c\right\} &\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\int_{u_1}^{u_1+h} Z_n^{1/4}(s,l)ds \geq \frac{h^{1/2}}{2}, \mathbf{B}^c\right\}\\
&+ \mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{v\in\delta_l}\int_{u_1}^{u_1+h}\left|Z_n^{1/4}(s,v) - Z_n^{1/4}(s,l)\right| ds \geq \frac{h^{1/2}}{2}, \mathbf{B}^c\right\}
\end{aligned} \tag{20}$$

Note that by (9), we have for any vector $u = (u_1, u_2)$, $\mathbf{E}_{\vartheta_0}\left|Z_n(u)\right| \leq 1$ and by Markov and Holder inequalities, we have

$$\begin{aligned}
\mathrm{P}_{\vartheta_0}^{(n)}\left\{\int_{u_1}^{u_1+h} Z_n^{1/4}(s,l)ds \geq \frac{h^{1/2}}{2}, \mathbf{B}^c\right\} &\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\int_{u_1}^{u_1+h} Z_n^{1/4}(s,l)ds \geq \frac{h^{1/2}}{2}\right\}\\
&\leq Ch^5
\end{aligned}$$

If we denote by $\mathcal{I}$ the last probability and

$$Y_{n,m}(y,v) = \int_m^y Z_n^{1/4}(s,v)ds$$

Then

$$\begin{aligned}\mathcal{I} &= \mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{v\in\delta_l}\int_{u_1}^{u_1+h}\left|Z_n^{1/4}(s,v)-Z_n^{1/4}(s,l)\right|ds \geq \frac{h^{1/2}}{2},\mathbf{B}^c\right\}\\ &\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{\substack{|u_1-u_2|<h\\|v_1-v_2|<h}}|Y_{n,m}(u_1,v_1)-Y_{n,m}(u_2,v_2)| \geq \frac{h^{1/2}}{2}\right\}\end{aligned}$$

We now check that $Y_n(\cdot,\cdot)$ verifies the conditions of Lemma 5.5 in Ibragimov and Khasminskii (1981). Using the Hölder inequality we can write

$$\begin{aligned}Y_{n,m}^4(u,v) &= \left(\int_m^u Z_n^{\frac{1}{4}}(s,v)ds\right)^4\\ &\leq \left(u-m\right)^3\int_m^u Z_n(s,v)ds\end{aligned}$$

Then because $u_1 \leq m+1$ and since $\mathbf{E}_{\vartheta_0}Z_n(s,v) \leq 1$ we obtain

$$\mathbf{E}_{\vartheta_0}Y_{n,m}^4(u,v) \leq (u-m)^4 \leq 1, \tag{21}$$

and

$$\begin{aligned}Y_{n,m}(u',v')-Y_{n,m}(u,v) &= \int_m^{u'} Z_n^{1/4}(s,v')ds - \int_m^u Z_n^{1/4}(s,v)ds\\ &= \left[\int_m^{u'} Z_n^{1/4}(s,v')ds - \int_m^{u'} Z_n^{1/4}(s,v)ds\right] + \left(\int_u^{u'} Z_n^{1/4}(s,v)ds\right)\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}_{\vartheta_0}\left|Y_{n,m}(u,v)-Y_{n,m}(u',v')\right|^4 &\leq 2^3\mathbf{E}_{\vartheta_0}\left|\int_m^{u'}\left[Z_n^{1/4}(s,v')-Z_n^{1/4}(s,v)\right]ds\right|^4\\ &\quad + 2^3\mathbf{E}_{\vartheta_0}\left|\left(\int_u^{u'} Z_n^{1/4}(s,v)ds\right)\right|^4\end{aligned}$$

Similarly to (21) we obtain:

$$\mathbf{E}_{\vartheta_0}\left|\left(\int_u^{u'} Z_n^{1/4}(s,v)ds\right)\right|^4 \leq C_{1*}|u'-u|^4$$

and

$$\mathbf{E}_{\vartheta_0}\left|\int_m^{u'}\left[Z_n^{1/4}(s,v')-Z_n^{1/4}(s,v)\right]ds\right|^4 \leq (u-m)^3\int_m^{u'}\mathbf{E}_{\vartheta_0}\left[Z_n^{1/4}(s,v')-Z_n^{1/4}(s,v)\right]^4 ds$$

Moreover, we can prove using Kutoyants (1998), (1.32), lemma 1.5 that

$$\mathbf{E}_{\vartheta_0}\left[Z_n^{1/4}(s,v')-Z_n^{1/4}(s,v)\right]^4 \leq R\,|v-v'|^4$$

Therefore we have

$$\mathbf{E}_{\vartheta_0}\left|Y_n(u,v)-Y_n(u',v')\right|^4 \leq C\left(|u-u'|^4+|v-v'|^4\right) \tag{22}$$

Then the conditions of Lemma 5.5 in Kutoyants (1998) are now verified for $Y_n(\cdot,\cdot)$ with $m=r=4$ and $d=2$. By Markov inequality and Lemma 5.5 we obtain

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{\substack{|u-u'|<h\\|v-v'|<h}}|Y_n(u,v)-Y_n(u',v')|\geq\frac{h^{1/2}}{2}\right\}\leq \mathbf{C}h^2.$$

Therefore

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{Y_h^m\geq h^{1/4}\right\}\leq \mathbf{C}h^{1/4} \tag{23}$$

By the same technique we can prove similar inequalities for all terms in $\Delta_h^{m,l}\left(Z_n^{\frac14}(u,v)\right)$. We can deduce, there exists a constant γ such that

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\Delta_h^{m,l}\left(Z_n^{\frac14}(u,v)\right)\geq h^{1/4}\right\}\leq\gamma h^{1/4}. \tag{24}$$

This inequality helps us to derive the estimate on the tails of the process $Z_n(\cdot)$ outside the squares Q_L.

Lemma 5. *There exist constant $c_1>0$, $c_2>0$ and C such that*

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{u,v\in\delta_{m,l}} Z_n(u,v)>e^{-c_1(|m|+|l|)}\right\}\leq Ce^{-c_2(|m|+|l|)} \tag{25}$$

Proof

Subdivise the square $\delta_{m,l}$ into M^2 of small squares $d_{i,j} = [u_i, u_{i+1}] \times [v_i, v_{i+1}]$ of length h (h was be defined below). Then

$$\begin{aligned}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{u,v\in\delta_{m,l}} Z_n(u,v) > e^{-c_1(|m|+|l|)}\right\} &\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{u,v\in\delta_{m,l}} Z_n^{1/4}(u,v) > e^{\frac{-c_1(|m|+|l|)}{4}}\right\} \\ &\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\max_{u_i,v_j\in d_{i,j}} Z_n^{1/4}(u_i,v_j) > \frac{1}{2}e^{\frac{-c_1(|m|+|l|)}{4}}\right\} \\ &\quad + \mathrm{P}_{\vartheta_0}^{(n)}\left\{\Delta_h^{m,l}\left(Z_n^{\frac{1}{4}}(u,v)\right) \geq \frac{1}{2}e^{\frac{-c_1(|m|+|l|)}{4}}\right\}\end{aligned}$$

We can majored the first probability by follow

$$\begin{aligned}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\max_{u_i,v_j\in d_{i,j}} Z_n^{1/4}(u_i,v_i) > \frac{1}{2}e^{\frac{-c_1(|m|+|l|)}{4}}\right\} &\leq \sum_{u_i,v_j\in d_{i,j}} \mathrm{P}_{\vartheta_0}^{(n)}\left\{Z_n^{1/4}(u_i,v_i) > \frac{1}{2}e^{\frac{-c_1(|m|+|l|)}{4}}\right\} \\ &\leq \sum_{u_i,v_j\in d_{i,j}} 4e^{\frac{c_1(|m|+|l|)}{2}}\mathbf{E}_{\vartheta_0} Z_n^{1/2}(u_i,v_i),\end{aligned}$$

and by (11) we obtain

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\max_{u_i,v_j\in d_{i,j}} Z_n^{1/4}(u_i,v_i) > \frac{1}{2}e^{\frac{-c_1(|m|+|l|)}{4}}\right\} \leq 4e^{\frac{c_1(|m|+|l|)}{2}} M^2 e^{-\kappa(|m|+|l|)}.$$

Now, put $M = 2^{1/4}\exp\left\{\frac{c_1}{4}(|m|+|l|)\right\}$ we have $h = \frac{1}{M}$. Then by (24) we have

$$\begin{aligned}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\Delta_{1/M}^{m,l}\left(Z_n^{\frac{1}{4}}(u,v)\right) \geq \frac{1}{2}e^{\frac{-c_1(|m|+|l|)}{4}}\right\} &= \mathrm{P}_{\vartheta_0}^{(n)}\left\{\Delta_h^{m,l}\left(Z_n^{\frac{1}{4}}(u,v)\right) \geq h^{1/4}\right\} \\ &\leq Ch^{1/4} = \frac{1}{2}C\exp\left\{\frac{c_1}{4}(|m|+|l|)\right\}\end{aligned}$$

Therefore we have

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{u,v\in\delta_{m,l}} Z_n(u,v) > e^{-c_1(|m|+|l|)}\right\} \leq Ce^{-c_2(|m|+|l|)}.$$

4.2. Consistency and convergence in law of the BE

Now we can begin to study the properties of the BE. We need firsly this lemma.

Lemma 6. *There exist constant $c > 0$ and $C > 0$ such that for $L > 0$*

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\int_{Q_L^c} uZ_n(u,v)dudv > e^{-cL}\right\} \le Ce^{-cL}. \tag{26}$$

Proof. Introduce the sets $\Gamma_k = Q_k \setminus Q_{k-1}$. Then each Γ_k contains $8k-4$ unit squares $\delta_{m,l}$ with $(m,l) \in \{-(k-1),(k-1)\} \times \{0,\pm1,\ldots,\pm k\}$ and $(m,l) \in \{0,\pm1,\ldots,\pm k\} \times \{-(k-1),(k-1)\}$

Since $e^{-cL} = (1-e^{-c})\sum_{k=L}^{\infty} e^{-ck}$ and put

$$\mathcal{T}_n = \mathrm{P}_{\vartheta_0}^{(n)}\left\{\int_{Q_L^c} uZ_n(u,v)dudv > e^{-cL}\right\}$$

we have

$$\begin{aligned}
\mathcal{T}_n &\le \mathrm{P}_{\vartheta_0}^{(n)}\left\{\sum_{k=L}^{\infty}\int_{\Gamma_k} uZ_n(u,v)dudv > (1-e^{-c})\sum_{k=L}^{\infty} e^{-ck}\right\}\\
&\le \sum_{k=L}^{\infty}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sum_{\delta_{m,l}\in\Gamma_k}\int_{\delta_{m,l}}(m+1)Z_n(u,v)dudv > (1-e^{-c})e^{-ck}\right\}\\
&\le \sum_{k=L}^{\infty}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sum_{\delta_{m,l}\in\Gamma_k}\int_{\delta_{m,l}}(m+1)Z_n(u,v)dudv > a_0e^{-ck}\right\}\\
&\le \sum_{k=L}^{\infty}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sum_{\delta_{m,l}\in\Gamma_k}\int_{\delta_{m,l}} Z_n(u,v)dudv > a_0e^{-ck-\ln(m+1)}\right\}\\
&\le \sum_{k=L}^{\infty}\sum_{\delta_{m,l}\in\Gamma_k}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{\delta_{m,l}} Z_n(u,v) > a_1e^{-ck-\ln(m+1)}\right\}\\
&\le \sum_{k=L}^{\infty} Ae^{-ck} \le Ce^{-cL}.
\end{aligned}$$

Therefore, we obtain the convergence in law of the estimator in (6).

4.3. Convergence of the moments of the BE

To prove the convergence of moments, we need several lemmas

Lemma 7. *There exists a constant $C > 0$ such that*

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\int\int_{\mathbb{R}^2} Z_n(u)du < \frac{h^2}{4}\right\} \le Ch^{1/2}$$

Proof. For all $h > 0$, we have

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\int\int_{\mathbb{R}^2} Z_n(u)du < \frac{h^2}{4}\right\} \leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\int_{-h}^{h}\int_{-h}^{h} Z_n(u)du < \frac{h^2}{4}\right\},$$

because

$$Z_n(0) = 1 \quad \text{then} \quad \int_{-h}^{h}\int_{-h}^{h} Z_n(u)du = 4h^2.$$

Therefore

$$\begin{aligned}
\mathrm{P}_{\vartheta_0}^{(n)}\left\{\int\int_{\mathbb{R}^2} Z_n(u)du < \frac{h^2}{4}\right\} &\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\int_{-h}^{h}\int_{-h}^{h} Z_n(u) - Z_n(0)du < -\frac{15}{4}h^2\right\} \\
&\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\int_{-h}^{h}\int_{-h}^{h} Z_n(u) - Z_n(0)du < -\frac{1}{4}h^2\right\} \\
&\leq \mathrm{P}_{\vartheta_0}^{(n)}\left\{\left|\int_{-h}^{h}\int_{-h}^{h} Z_n(u) - Z_n(0)du\right| > \frac{1}{4}h^2\right\} \\
&\leq \frac{4}{h^2}\int_{-h}^{h}\int_{-h}^{h} \mathbf{E}_{\vartheta_0}\left|Z_n(u) - Z_n(0)\right| du
\end{aligned}$$

Using Cauchy inequality and $\mathbf{E}_{\vartheta_0} Z_n(u) \leq 1$, we have

$$\begin{aligned}
&\mathbf{E}_{\vartheta_0}\left|Z_n(u) - Z_n(0)\right| \\
&\leq \mathbf{E}_{\vartheta_0}\left[\left|Z_n^{1/2}(u) - Z_n^{1/2}(0)\right|^2\right]^{1/2}\left[\mathbf{E}_{\vartheta_0}\left|Z_n^{1/2}(u) + Z_n^{1/2}(0)\right|^2\right]^{1/2} \\
&\leq \left[2\mathbf{E}_{\vartheta_0}\left|Z_n(u)\right| + 2\mathbf{E}_{\vartheta_0}\left|Z_n(0)\right|\right]^{1/2}\left[\mathbf{E}_{\vartheta_0}\left|Z_n^{1/2}(u) - Z_n^{1/2}(0)\right|^2\right]^{1/2} \\
&\leq c||u||^{1/2} \qquad \text{by (2)} \\
&\leq Ch^{1/2}
\end{aligned}$$

Then

$$\mathrm{P}_{\vartheta_0}^{(n)}\left\{\int\int_{\mathbb{R}^2} Z_n(u)du < \frac{h^2}{4}\right\} \leq Ch^{1/2}$$

Lemma 8. *There exists a constant* α *and* κ *such that*

$$\mathbf{E}_{\vartheta_0}\left(\int\int_{\delta_{m,l}} \frac{Z_n(u)du}{\int\int_{\mathbb{R}^2} Z_n(u)du}\right) \le \kappa e^{-\alpha(|m|+|l|)}$$

Proof. By Lemma 5 we obtain

$$\begin{aligned}\mathrm{P}_{\vartheta_0}^{(n)}\left\{\int\int_{\delta_{m,l}} Z_n(u)du > e^{-\beta(|m|+|l|)}\right\} &\le \mathrm{P}_{\vartheta_0}^{(n)}\left\{\sup_{\delta_{m,l}} Z_n(u) > e^{\beta(|m|+|l|)}\right\} \\ &\le Ce^{-c_2(|m|+|l|)}\end{aligned}$$

We put

$$\mathcal{I}_n = \int\int_{\delta_{m,l}} \frac{Z_n(u)du}{\int\int_{\mathbb{R}^2} Z_n(v)dv}, \qquad \mathcal{B}_n = \left\{\omega : \int\int_{\mathcal{R}^2} Z_n(u)du < \frac{h^2}{4}\right\},$$
$$\mathcal{D}_n = \left\{\omega : \int\int_{\delta_{m,l}} Z_n(u)du > e^{-\beta(|m|+|l|)}\right\}$$

Since $\mathcal{I}_n \le 1$. We have

$$\begin{aligned}\mathbf{E}_{\vartheta_0}[\mathcal{I}_n] &= \mathbf{E}_{\vartheta_0}\left[\mathcal{I}_n\mathbb{I}_{\mathcal{B}_n}\right] + \mathbf{E}_{\vartheta_0}\left[\mathcal{I}_n\mathbb{I}_{\mathcal{B}_n^c}\right] \\ &\le \mathbf{E}_{\vartheta_0}\left[\mathbb{I}_{\mathcal{B}_n}\right] + \mathbf{E}_{\vartheta_0}\left[\mathcal{I}_n\mathbb{I}_{\mathcal{B}_n^c}\mathbb{I}_{\mathcal{D}_n}\right] + \mathbf{E}_{\vartheta_0}\left[\mathcal{I}_n\mathbb{I}_{\mathcal{B}_n^c}\mathbb{I}_{\mathcal{D}_n^c}\right] \\ &\le \mathbf{E}_{\vartheta_0}\left[\mathbb{I}_{\mathcal{B}_n}\right] + \mathbf{E}_{\vartheta_0}\left[\mathbb{I}_{\mathcal{B}_n^c\cap\mathcal{D}_n}\right] + \mathbf{E}_{\vartheta_0}\left[\mathcal{I}_n\mathbb{I}_{\mathcal{B}_n^c\cap\mathcal{D}_n^c}\right]\end{aligned}$$

For $\omega \in \mathcal{B}_n^c \cap \mathcal{D}_n^c$

$$\mathcal{I}_n \le \frac{4}{h^2}e^{-\beta(|m|+|l|)}$$

Then

$$\begin{aligned}\mathbf{E}_{\vartheta_0}[\mathcal{I}_n] &\le \mathbf{E}_{\vartheta_0}\left[\mathbb{I}_{\mathcal{B}_n}\right] + \mathbf{E}_{\vartheta_0}\left[\mathbb{I}_{\mathcal{B}_n^c\cap\mathcal{D}_n}\right] + \frac{4}{h^2}e^{-\beta(|m|+|l|)}\mathbf{E}_{\vartheta_0}\left[\mathbb{I}_{\mathcal{B}_n^c\cap\mathcal{D}_n^c}\right] \\ &\le \mathrm{P}_{\vartheta_0}^{(n)}\{\mathcal{B}_n\} + \mathrm{P}_{\vartheta_0}^{(n)}\{\mathcal{B}_n^c \cap \mathcal{D}_n\} + \frac{4}{h^2}e^{-\beta(|m|+|l|)} \\ &\le \mathrm{P}_{\vartheta_0}^{(n)}\{\mathcal{B}_n\} + \mathrm{P}_{\vartheta_0}^{(n)}\{\mathcal{D}_n\} + \frac{4}{h^2}e^{-\beta(|m|+|l|)} \\ &\le Ch^{1/2} + Ce^{-c_2(|m|+|l|)} + \frac{4}{h^2}e^{-\beta(|m|+|l|)}\end{aligned}$$

If we put $h^2 = e^{-\frac{\beta}{2}(|m|+|l|)}$, we obtain

$$\mathbf{E}_{\vartheta_0}[\mathcal{I}_n] \leq \kappa e^{-\alpha(|m|+|l|)},$$

which finishes the proof of lemma. To prove the convergence of the moments, it is enough to prove the uniform integrability of the moments of the Bayesian estimator. By Hölder inequality we have

$$\begin{aligned}\int\int_{\mathbb{R}^2} u_1 Z_n(u)du &= \int\int_{\mathbb{R}^2} u_1 \left[Z_n(u)\right]^{1/p} \left[Z_n(u)\right]^{1/q} du \\ &\leq \left(\int\int_{\mathbb{R}^2} u_1^p \left[Z_n(u)\right] du\right)^{1/p} \left(\int\int_{\mathbb{R}^2} \left[Z_n(v)\right] dv\right)^{1/q}\end{aligned}$$

here we put $u = (u_1, u_2)$.

$$\begin{aligned}\mathbf{E}_{\vartheta_0}\left|\int\int_{\mathbb{R}^2} \frac{u_1 Z_n(u)du}{\int\int_{\mathbb{R}^2} Z_n(v)du}\right|^p &\leq \mathbf{E}_{\vartheta_0}\left|\int\int_{\mathbb{R}^2} \frac{|u_1|^p Z_n(u)du}{\int\int_{\mathbb{R}^2} Z_n(v)dv}\right| \\ &\leq \mathcal{C}\sum_{m=0}^{\infty}(m+1)^p \mathbf{E}_{\vartheta_0}\left(\int\int_{\delta_{m,l}} \frac{Z_n(u)du}{\int\int_{\mathbb{R}^2} Z_n(v)dv}\right) \\ &\leq \mathcal{H}\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\mathbf{E}_{\vartheta_0}\left\|n\left(\tilde{\vartheta}_n - \vartheta_0\right)\right\|^p &\leq \mathcal{K}_1 \mathbf{E}_{\vartheta_0}\left|\int\int_{\mathbb{R}^2} \frac{u_1 Z_n(u)du}{\int\int_{\mathbb{R}^2} Z_n(v)dv}\right|^p + \mathcal{K}_2 \mathbf{E}_{\vartheta_0}\left|\int\int_{\mathbb{R}^2} \frac{u_2 Z_n(u)du}{\int\int_{\mathbb{R}^2} Z_n(v)dv}\right|^p \\ &\leq \mathcal{K},\end{aligned}$$

and we have convergence of all polynomial moments.

5. Conclusion

We studied the properties of the Bayesian estimator of the position of a source emitting a Poisson signal. We were interested in the case where the intensity function of this process is discontinuous and misspecified in the sense of (4) while the true intensity is describe in (3) where $\lambda(t), t \geq 0$ is not exactly and it bounded below by a constant value $\lambda_1 > 0$. Then for these typical case which is particularity of the class of the wrong modele gives us a model of Poisson process whose distribution is the closest within the given parametric family of distributions. More precisely the "pseudo" BE finds a model in this family which fits better to the real model than any other, using the Kullback-Leibner divergence criteria. This is how we have shown the consistency, the convergence in distribution of this "pseudo" Bayesian estimator and its convergence of moments because for the class of intensity in (4), the estimator $\hat{\vartheta}_0$ which minimize the Kullback-Leibner divergence coincide with the true value of the parameter ϑ_0 of the real intensity in (3).

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